1 Points in space

Welcome to multivariable calculus! The two core ideas of the calculus,

- linear approximation and
- break things into tiny pieces and add them together,

should be familiar to you. So the main challenge of this class will be to

- visualize the above happening in two and three dimensional space and
- to understand Stokes' Theorem.

Let us now begin at the beginning.

A *real number* is a point on the real number line, $\mathbb{R}$. (We won't worry about what that means...) A point of *n-space*, $\mathbb{R}^n$, is an ordered list of n real numbers. Eg

$$(1), \qquad (1, 2), \qquad (1, 2, 3)$$

are examples of points in 1-, 2-, and 3-space, respectively. We *think* of points as being locations, or addresses. We'll use lower case letters to denote points, writing equations like $p = (1, 2, 3)$ and $q = (1, 2)$.

Upper case letters are used to denote *sets* of points. For example, we could take $O = \{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)\}$. As a short hand for "plus or minus one" we'll write ± 1. So the set O becomes

$$O = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

As another example we could take

$$C = \left\{ \left(\frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}} \right) \right\}.$$

Note that C contains eight points, not just two.

It is somewhat tedious to actually *list* all the points of a set, *especially* when the set is infinite. So we resort to the well-known *set-builder notation*: we take $\{\text{foo} \mid \text{bar}\}$ to mean "the set of all points *foo* which satisfy the condition *bar*." Some canonical examples are

$$X = \{p \in \mathbb{R}^3 \mid \text{there is an } x \in \mathbb{R} \text{ with } p = (x, 0, 0)\},$$

$$Y = \{p \in \mathbb{R}^3 \mid \text{there is an } y \in \mathbb{R} \text{ with } p = (0, y, 0)\},$$

$$Z = \{p \in \mathbb{R}^3 \mid \text{there is an } z \in \mathbb{R} \text{ with } p = (0, 0, z)\},$$

The phrase $p \in \mathbb{R}^3$ is pronounced "p is in 3-space." The three sets X, Y, and Z are called the *coordinate axes*. For a picture of what X, Y, and Z look like in $\mathbb{R}^3$ you should just stare at the corner of the room where two walls meet the ceiling. Or see Figure 1, below.

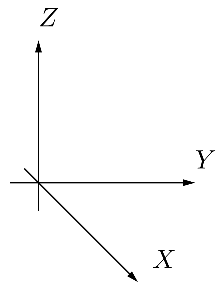

Figure 1: A perspective view of the three coordinate axes.

The arrows of the axes indicate the "positive" direction. We think of X as coming out of the page and of Y and Z as lying in the page. We rotate Y counter-clockwise to get it to lie on top of Z. The point where they all intersect is called, poetically, the *origin*. It is a bit of a challenge to remember how the axes are arranged in 3-space. For a mnemonic we use the *right-hand rule*: point the fingers of your right hand in the positive x-direction and curl your fingers towards the positive y direction. Your thumb should now point towards the positive z direction. To be totally clear: to check the right-hand rule you need an ordered collection of three oriented lines. If the right hand rule holds then we call the collection of lines *positively oriented*. If the right hand rule does not hold (the "left hand rule") then the collection of lines is *negatively oriented*.

Exercise 1.1. Suppose we move the coordinate axes so the x-axis points towards your feet and the z-axis bursts out of your chest. Is your right or left arm pointing in the positive x-direction?

Exercise 1.2. There are six possible orderings on the coordinate axes: (X, Y, Z), (X, Z, Y), etc. List them. How many are positively oriented? How many are negatively oriented?

As another, slightly less clear, example of the set-builder notation we have the *coordinate planes*:

$$XY = \{(x, y, 0) \mid x \in \mathbb{R}, y \in \mathbb{R}\},$$

$$YZ = \{(0, y, z) \mid y \in \mathbb{R}, z \in \mathbb{R}\},$$

$$ZX = \{(x, 0, z) \mid z \in \mathbb{R}, x \in \mathbb{R}\}.$$

Figure 2 shows a bit of the three coordinate planes together with the axes.

2 Vectors

Now that we have points in space it would nice if we had a way to move between them. That is, given p and q in $\mathbb{R}^3$ we want a set of points in $\mathbb{R}^3$ which "connects"

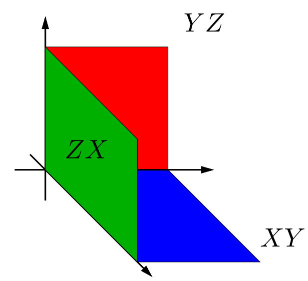

Figure 2: What would the picture look like if you rotated it a bit, around the z-axis?

one to the other. Here is an attempt. For every $t \in \mathbb{R}$ let $L(t) = p + t(q - p)$. Then $L(0) = p$ and $L(1) = q$. Very neat. There is one little problem here: subtracting points makes no sense! (Can you subtract Oregon from California? What would be left over?)

So we *make* it make sense: Define a difference of points, $v = q - p$, to be a *vector*. We call p the *base point* of v. Sometimes p is also called the *initial* point (or tail) of v while q is called the *terminal* point (or head) of v. To emphasize the base point, we sometimes write v_p instead of just v.

Unlike points, vectors can be added and scaled. Just like points, a vector based at p is determined by three real numbers. Here is the difference: a point is a *location* in space, while a vector represents a *motion* through space; a direction together with a magnitude.

Here is an example: If you are standing at $(1, 1)$ in $\mathbb{R}^2$ and you little brother comes and pushes you toward the x-axis then your position is still $(1, 1)$ but you feel a force pushing you in the direction $(0, -1)$. If your little sister comes and starts pushing you toward the y-axis (in the $(-5, 2)$ direction, say) then you will feel a net force vaguely towards the point $(0, 0)$, the exact direction depending on the ratio of how hard they are shoving you.

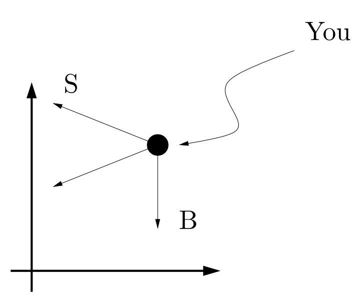

Figure 3: You are the big dot.

As a bit of notation, if we are given a vector $v = (x, y, z)$ we say that x is the first component of v, y is the second component, etc. The result of the above discussion is that vectors, based at the same point, add component-wise:

$$(x, y, z)_p + (x', y', z')_p = (x + x', y + y', z + z')_p.$$

It is also possible to *scale* a vector – if r is a real number we have:

$$r(x, y, z)_p = (rx, ry, rz)_p.$$

If v_p and w_p are non-zero scalar multiples of each other, we call them *parallel*. Finally, it is possible to add a vector to a point and get a new point. But v_p can *only* be added to p itself:

$$(x, y, z)_{(a,b,c)} + (a, b, c) = (x + a, y + b, z + c).$$

With these definitions, our formula $L(t) = p + t(q - p)$ makes sense. We form a vector $(q - p)_p$, we scale it by a factor of t, and we add the resulting vector (still based at p!) back to the point p.

Exercise 2.1. Describe the line in $\mathbb{R}^2$ going through the points $q = (1, 1)$ and $p = (1, 0)$. Draw a sketch.

Exercise 2.2. Describe the line in $\mathbb{R}^3$ going through the points $q = (1, 1, 1)$ and $p = (0, 0, 0)$. Draw a sketch.

Exercise 2.3. Our formula for a line connecting p and q is so simple one is tempted to generalize it. Describe the plane $P(s, t)$ going through a triple of points q, q', and p.

Next, specialize to the case $q = (0, 1, 0)$, $q' = (0, 0, 1)$, and $p = (1, 0, 0)$. Draw a sketch.

We end this discussion of vectors by mentioning a few "special" vectors. Since $\mathbb{R}^3$ consists of points with three coordinates, at every point there are three directions which are naturally "picked out". They are

$$\mathbf{i} = (1, 0, 0),$$
$$\mathbf{j} = (0, 1, 0),$$
$$\mathbf{k} = (0, 0, 1).$$

Any vector is the sum of these three. For example, if $v = (1, 2, 3)$ then $v = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. In general, if $v = (x, y, z)$ then $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Here are a few more exercises, just for fun.

Exercise 2.4. Draw a few pictures to show that the set O equals the vertices of a regular octahedron. What does C represent?

Exercise 2.5. (Challenging) Find the coordinates of the vertices of a regular tetrahedron. You should assume that one vertex lies at $(1, 0, 0)$ and one vertex lies in the xz-plane