1 Dot product

Suppose that $v = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$. We need to define the length of v. There is no way to deduce the length from first principles. We instead make this our main definition:

Definition 1.1. The *norm* of $v = (v_1, v_2, \ldots, v_n)$ is defined to be

$$|v| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

So in $\mathbb{R}^2$, if $v = (x, y)$ then $|v| = \sqrt{x^2 + y^2}$. In $\mathbb{R}^3$, the vector $v = (x, y, z)$ has norm $|v| = \sqrt{x^2 + y^2 + z^2}$. The *distance* between two points $p, q \in \mathbb{R}^n$ is then the norm (or *length*) of the vector $q - p$. The norm is also sometimes called the *magnitude* of the vector.

Exercise 1.2. Recall the sets $O = \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ and C defined in the 9/01 notes – the vertices of the cube and octahedron. What are the lengths of the edges of the cube and of the octahedron?

If v and w are both vectors it is naturally of interest to relate the length of $v + w$ and $v - w$ to the lengths of v and w. So consider $|v - w|^2$, as it is always nice to avoid unnecessary square roots:

$$|v - w|^2 = (v_1 - w_1)^2 + (v_2 - w_2)^2 + \ldots + (v_n - w_n)^2.$$

Multiplying out the right hand side we find:

$$|v - w|^2 = v_1^2 - 2v_1w_1 + w_1^2 + \ldots + v_n^2 - 2v_nw_n + w_n^2.$$

Rearranging the terms we find that

$$|v - w|^2 = |v|^2 - 2(v_1w_1 + \ldots + v_nw_n) + |w|^2.$$

That is, $|v-w|^2$ is *almost* equal to $|v|^2+|w|^2$. The only obstruction is the "error term" $v_1w_1 + \ldots + v_nw_n$. We give this error term a name:

Definition 1.3. The *dot product* of two vectors v and w is

$$v \cdot w = v_1w_1 + \ldots + v_nw_n.$$

With this notation we have $|v - w|^2 = |v|^2 + |w|^2 - 2v \cdot w$. (Can you produce a similar formula for $|v + w|^2$?) It follows that the dot product is zero if and only if the "Pythagorean Theorem" holds for the three vectors v, w, and $v - w$. See Figure 1.

As a bit of terminology we say that v and w are *orthogonal* if $v \cdot w = 0$.

Exercise 1.4. Check that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all orthogonal to each other.

The dot product has many nice properties and the most important of these is $v \cdot v = |v|^2$. (Again we see that the square of the norm is algebraically more natural than the norm itself.) Also, if w is parallel to v, say $w = rv$ then we have $v \cdot w = v \cdot (rv) = rv \cdot v = r|v|^2$.

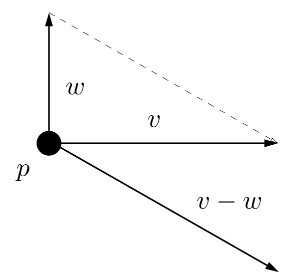

Figure 1: The dotted line is parallel to the vector $v - w$.

Exercise 1.5. Verify the *parallelogram law*: $|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2$. Explain *what* the law has to do with parallelograms.

To repeat, if v and w are orthogonal then the dot product is zero and if v and w are parallel then the dot product is a multiple of $|v|^2$. Thus in some sense the dot product measures how much v and w "point in the same direction". We can eventually make this precise, as follows: if $|v| = 1$ we call v a *unit vector*. Note that for any non-zero vector w the parallel vector $w/|w|$ is always a unit vector.

Definition 1.6. The unit sphere in $\mathbb{R}^n$ is the set $S^{n-1} = \{p \in \mathbb{R}^n \mid |p-0| = 1\}$. Here $0 = (0, 0, \ldots, 0)$ is the *origin* of $\mathbb{R}^n$. Another way to write this set is $S^{n-1} = \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$. Yet another way to denote this set will be $S^{n-1}\colon x_1^2 + x_2^2 + \ldots + x_n^2 - 1$.

Moral: The point of introducing the unit sphere is that it perfectly records all possible *directions* of vectors while forgetting all information about their norm.

Exercise 1.7. Draw a picture of $S^1 \subset \mathbb{R}^2$. Draw a picture of $S^2 \subset \mathbb{R}^3$. Verify that all points of O and C (defined in the 9/01 notes) lie in S^2, the unit sphere in $\mathbb{R}^3$. Explain why we use the notation S^2 for the sphere in $\mathbb{R}^3$ *instead* of using the notation S^3.

Here is a nice fact, called the Cauchy-Schwartz Inequality:

Theorem 1.8. *For any pair of vectors v and w we have $|v \cdot w| \leq |v||w|$, with equality holding if and only if v and w are parallel.* □

In particular, if v, w are unit vectors then deduce that $-1 \leq v \cdot w \leq 1$. Since this is exactly the set of possible values of cosine we can *define* the angle between the vectors to be $\theta = \arccos(v \cdot w)$. It is easier to remember this definition in the form $\cos(\theta) = v \cdot w$: see Figure 2.

Now, if v and w are *any* non-zero vectors then $v/|v|$ and $w/|w|$ are unit vectors, and we now define the angle between v and w to be

$$\theta = \arccos\left(\frac{v}{|v|} \cdot \frac{w}{|w|}\right) = \arccos\left(\frac{v \cdot w}{|v||w|}\right).$$

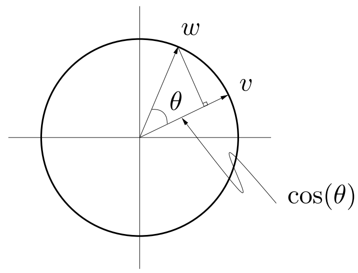

Figure 2: The distance from the origin to the foot of the perpendicular is $\cos(\theta)$.

This follows the moral set out above: the direction of the vector is captured by its parallel unit vector. Again, it is probably easier to remember the formula $v \cdot w = |v||w|\cos(\theta)$. This is sufficiently intricate to require an exercise or two.

Exercise 1.9. Compute the angles between the vectors with initial point the origin and with terminal points at the vertices of the octahedron. (This is very easy.) Do the same for the cube. (This is less easy!)

We now have enough tools to do an honest calculus problem. Here is the question. Suppose that p and q are points in $\mathbb{R}^3$ and suppose that $L(t) = t(q-0)$ is the line connecting the origin to q. What is the closest point of L to p? Another way to say this is: what is the *closest point projection* of p to L?

Clearly this is an optimization problem. The way to solve the problem is to compute the distance from p to $L(t)$, take the derivative with respect to t, set equal to zero, and solve. (Hint: if you actually do this, don't differentiate the distance, but rather the distance squared – the algebra will be nicer.) However, this is not a one variable calculus class.

Instead let us argue as follows: suppose that $L(t)$ is the desired closest point. Connect p to $L(t)$ by a straight line M, as that is the shortest distance between two points. The angle between L and M is a right angle – otherwise we could move the foot of M slightly in one direction or the other and decrease the length of M. Thus the desired point $L(t)$ is found by forming a right triangle with $p-0$ as the hypotenuse, M as one leg, and $L(t)-0$ as the other leg. Now, let $v = p-0$, $w = q-0$, and $u = L(t)-0$. (It may help here to draw a picture showing p, q, 0, v, w, and u.) Now, u and w are parallel. So the angle between v and w is the same as the angle between v and u. Calling this angle θ we have $\cos(\theta) = \frac{v \cdot w}{|v||w|}$. Deduce, perhaps by lookinf at your picture, that $|u| = |v|\cos(\theta)$. Since u has the same direction as w we have $u = \frac{|v|\cos(\theta)}{|w|} w = \frac{|v||w|\cos(\theta)}{|w|^2} w = \frac{v \cdot w}{w \cdot w} w$. This removes any mention of t, and so solves the problem. The book calls the vector

$$u = \mathrm{proj}_w(v) = \frac{v \cdot w}{w \cdot w} w$$

the *vector projection* of v to w.

2 Cross product

We do not actually have enough time in the class to discuss determinants in any depth. So we will simply assert a few facts.

Suppose that u, v, and w are vectors in $\mathbb{R}^3$, all based at the origin, say. Then these vectors span a *parallelepiped*: the set $P(u,v,w) = \{ru+sv+tw \mid r,s,t \in [0,1]\}$. See Figure 3.

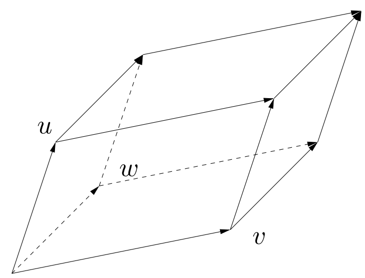

Figure 3: What happens when two of the vectors are parallel?

Suppose that $u = (u_{\mathbf{i}}, u_{\mathbf{j}}, u_{\mathbf{k}})$, $v = (v_{\mathbf{i}}, v_{\mathbf{j}}, v_{\mathbf{k}})$, and $w = (w_{\mathbf{i}}, w_{\mathbf{j}}, w_{\mathbf{k}})$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{j}$ are the usual coordinate directions. Then following is an amazing formula for the volume of $P(u,v,w)$:

$$\mathrm{vol}(P) = u_{\mathbf{i}}v_{\mathbf{j}}w_{\mathbf{k}} - u_{\mathbf{i}}v_{\mathbf{k}}w_{\mathbf{j}} + u_{\mathbf{j}}v_{\mathbf{k}}w_{\mathbf{i}} - u_{\mathbf{j}}v_{\mathbf{i}}w_{\mathbf{k}} + u_{\mathbf{k}}v_{\mathbf{i}}w_{\mathbf{j}} - u_{\mathbf{k}}v_{\mathbf{j}}w_{\mathbf{i}}.$$

If we have the courage to rearrange this, pulling the u terms out, we find

$$\mathrm{vol}(P) = u_{\mathbf{i}}(v_{\mathbf{j}}w_{\mathbf{k}} - v_{\mathbf{k}}w_{\mathbf{j}}) + u_{\mathbf{j}}(v_{\mathbf{k}}w_{\mathbf{i}} - v_{\mathbf{i}}w_{\mathbf{k}}) + u_{\mathbf{k}}(v_{\mathbf{i}}w_{\mathbf{j}} - v_{\mathbf{j}}w_{\mathbf{i}})$$

and this can be written as a dot product:

$$u \cdot (v_{\mathbf{j}}w_{\mathbf{k}} - v_{\mathbf{k}}w_{\mathbf{j}}, v_{\mathbf{k}}w_{\mathbf{i}} - v_{\mathbf{i}}w_{\mathbf{k}}, v_{\mathbf{i}}w_{\mathbf{j}} - v_{\mathbf{j}}w_{\mathbf{i}}).$$

In a final fit of insanity, we *define* a new kind of product, the *cross product*, to record the second vector of the dot product above:

$$v \times w = (v_{\mathbf{j}}w_{\mathbf{k}} - v_{\mathbf{k}}w_{\mathbf{j}}, v_{\mathbf{k}}w_{\mathbf{i}} - v_{\mathbf{i}}w_{\mathbf{k}}, v_{\mathbf{i}}w_{\mathbf{j}} - v_{\mathbf{j}}w_{\mathbf{i}}).$$

At this point the book makes many claims. Such as:

- $\mathrm{vol}(P) = u \cdot (v \times w)$,
- the vector $v \times w$ is orthogonal to both v and w,
- the norm $|v \times w|$ is the area of the parallelogram spanned by v and w, and
- $|v \times w| = |v||w|\sin(\theta)$, where θ is the angle between v and w, as above.

To properly discuss these topics requires some linear algebra. We won't go any farther than this.