1 Describing sets

Here is a set of points:

$$A = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ x^2 + y^2 + z^2 = 1\}.$$

The description of the set is *implicit*: for every point $(x, y, z) \in \mathbb{R}^3$ we can apply the equation to decide whether or not the point is in the set. For example, $(1, 0, \frac{1}{2})$ is not in A as $1 + \frac{1}{4} > 1$. However, it is not clear how we can actually *find* a point that lies in A.

Here is another set:

$$B = \{(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \ \mid \ \theta \in [0, 2\pi], \phi \in [0, \pi]\}.$$

The description of B is *explicit*: a recipe is given for generating points in the set. However, given a point $(x, y, z) \in \mathbb{R}^3$ it is not clear how to determine if it lies in the set!

It is somewhat reassuring to note that $A = B$. Both sets describe the unit two-sphere S^2. It is somewhat unsettling that one set can be described in many ways. Can we list the points of a set given implicitly? Can we decide if a given point belongs to a set given explicitly? Can we decide if two descriptions give the same set or not?

Exercise 1.1. Suppose that $C = \{(x, y, z) \ \mid \ x + y + z = 0 \text{ and } z = (x - y)^2\}$. That is, C is the set of points (x, y, z) in $\mathbb{R}^3$ which satisfy both of the conditions $x + y + z = 1$ *and* $z = (x - y)^2$. Prove that the point $(\frac{1}{2}, \frac{1}{2}, 0)$ is in C. Now find *any* other point in C. What does the set C look like? Give a sketch. Can you describe C explicitly?

2 In dimension two

As a quick review, lets recall the situation in $\mathbb{R}^2$. We begin with sets described implicitly. For example, we have $L : y = mx+b$. So L is the set of points (x, y) satisfying $y = mx+b$. (Here m and b are fixed constants.) This is a line with slope m and y-intercept b. Not all lines can be described in this form – the remainder are vertical lines and we describe them via $L : x = c$.

Also familiar are the *conic sections*.

Circle – $S^1 : x^2 + y^2 = 1$ is the unit circle. It is the only conic section with a rotation symmetry

Ellipse – $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with axes of length $2a$ and $2b$. If $a > b$ then the *major axis* of E is the segment connecting the points $(\pm a, 0)$. The *minor axis* of E is then the segment collecting the points $(0, \pm b)$. We define the quantity $e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$ to be the *eccentricity* of E. Thus circles may be regarded as ellipses of eccentricity 0.

Parabola – $P : y = x^2$. Equally well, we could take $P' : x = y^2$ or $P'' : x + y = (x - y)^2$ etc.

Hyperbola – $H : x^2 - y^2 = 1$. Again, it is not a good idea to rely completely on these particular implicit forms – for example $H' : xy = 1$ is again a hyperbola. H and H' differ only by a $\pi/2$ rotation.

All of these are special cases of the *general quadratic*: the set of points (x, y) satisfying the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, qualitatively speaking, no matter the choices of A, B, C, etc the resulting curve always "looks like" one of the examples already given after rotation, translation, and/or rescaling of the coordinate plane.

Exercise 2.1. Create a movie (say, using **Maple**) of all of the hyperbolas $H_t : x^2-y^2 = t$ as t ranges between -10 and 10. What happens when $t = 0$? Make a movie of the parabolas $P_t : tx^2 = y$ for t is the same range. What happens at $t = 0$?

Of course, these sets can also be given explicitly. There are several ways to do this. The best known is $L : (t, mt + b)$ or in point/vector form $L : (0, b) + t(1, m)$. Also well known is $S^1 = \{(x, \pm\sqrt{1 - x^2}) \mid x \in [-1, 1]\}$. As always the square root causes difficulty – this description of S^1 is really a description of the upper and lower semicircles "glued together". We can cure this defect by noting that $S^1 = \{(\cos(\theta), \sin(\theta)) \mid \theta \in [0, 2\pi]\}$. Now we have removed the arbitrary separation of S^1 into "top" and "bottom" at the cost of introducing non-arithmetic functions, cosine and sine.

Exercise 2.2. Give explicit descriptions of E, P, and H. Avoid square roots as much as possible. (Hint for H: a "hyperbolic" version of cosine and sine will be useful.)

3 Geometric descriptions

We end our discussion of sets in dimension two by recalling the many possible *geometric* descriptions of sets. For example, a straight line is the shortest distance between two points. Alternatively, a line is obtained by taking all vectors (based at $(0, 0)$) which have dot product equal to A with a fixed *normal vector*, $\mathbf{n}$. We then write $L = \{p \in \mathbb{R}^2 \mid \mathbf{n} \cdot (p - 0) = A\}$. As an example we take $L : -mx + y = b$ and rewrite to obtain $L : (-m, 1) \cdot (x, y) = b$ of the desired form.

The unit circle S^1 is the set of points at distance one from the origin. An ellipse has two foci – if you take a point on the ellipse and add its distances to the foci the result is constant. The parabola is the set of points equidistant from a point and a line. The hyperbola also has two foci – if you take a point of the hyperbola and *subtract* its distances to the foci the result is constant.

Exercise 3.1. Fix a point $p \in \mathbb{R}^2$. Describe the set $D = \{q \in \mathbb{R}^2 \mid |q - p| \leq 1\}$. Also, describe the set $\{q\}$ so that $|q - 0| \leq 1$ and $|q - (1, 0)| \leq 1$ and $|q - (1/2, \sqrt{3}/2)| \leq 1$.

Exercise 3.2. Fix a number $a \in \mathbb{R}$. Find an explicit description of the parabola which is equidistant from the point $(a, 0)$ and from the line $L(t) = (-a, t)$.

4 In dimension three

We now tackle sets in dimension three. Again, the simplest implicit example is linear. Choose constants A, B, C, and D and let P be the set of points satisfying $Ax+By+Cz = D$. Equivalently, we could choose a *normal vector* $\mathbf{n} = (A, B, C)$ and a real number D. Then we can describe the plane P to be the set of points p so that $\mathbf{n} \cdot (p - 0) = D$.

As in dimension two, we have the general quadratic:

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + L = 0.$$

This now implicitly describes a *quadric* surface. Again, after rotation, translation, and rescaling of the coordinates we may assume that any quadric surface is one of the following:

Sphere – the points of the unit sphere S^2 solve $x^2+y^2+z^2 = 1$. In general, the sphere of radius R centered at (x_0, y_0, z_0) has equation $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$. There is an obvious connection with the set of vectors, based at (x_0, y_0, z_0), of length R. One way to obtain the sphere is to rotate the unit sphere in the xz plane about the z axis.

Exercise 4.1. Before reading on sketch pictures of the sets obtained by spinning the standard line, ellipse, parabola, and hyperbolas (there are two!) about the z axis. To be precise, spin the sets in the xz plane given by $z = x$, $2z^2 + x^2 = 1$, $z = x^2$, and $z^2 - x^2 = \pm 1$.

All of these "spun" sets have a rotation symmetry. This symmetry need not be present in a general quadric surface!

Ellipsoid – $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. In general we order the axes of the ellipsoid by length. The ellipsoid is obtained by spinning if and only if two of its axes have the same length.

Paraboloid – $P : \frac{x^2}{a^2} + \frac{y^2}{b^2} = z$.

Hyperbolic paraboloid – $HP : \frac{x^2}{a^2} - \frac{y^2}{b^2} = z$. The set HP cannot be obtained by spinning, even if $a = b$. The behaviour of HP near the origin deserves close inspection.

Hyperboloid – $H : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$. Careful: the sign of the right hand side is quite important! The two possibilities give the hyperboloid of one or two sheets, respectively.

Exercise 4.2. It is possible to go between geometric and implicit descriptions of the various spun quadric surfaces. Here is a more interesting challenge: Let $L(t) = (t, 0, -1)$ and $M(s) = (0, s, 1)$ be two lines in $\mathbb{R}^3$. Let W be the set of points in $\mathbb{R}^3$ equidistant from the lines L and M: for every point $p = (x, y, z)$ in W the distance from p to L equals the distance from p to M. Draw a sketch of W. Give an implicit description of W.

5 Cross sections

It is not so easy to understand a surface in $\mathbb{R}^3$. Let us end these notes by discussing the important notion of *cross sections*. In complete generality, suppose that S is a subset of $\mathbb{R}^n$. Let H be a *hyperplane* in $\mathbb{R}^n$: every $p \in H$ satisfies the equation $\mathbf{n} \cdot (p - 0) = A$ where $\mathbf{n}$ is a unit vector *normal* to H and A is the *height* of H. We call the intersection of the set S with H a *cross section*.

To make things less general, here is a simple example. Fix S to be the unit sphere in $\mathbb{R}^3$. Let $H_t = \{(x, y, t)\}$ be the plane parallel to the xy coordinate plane and containing the point $(0, 0, t)$. When t is very negative, H_t misses S. As we increase t we reach $t = -1$ and find an intersection of a single point. Increasing t from -1 to 0 causes the point to first turn into a small circle which then grows in size. At $t = 0$ the circle is as large as it gets, giving a unit circle in H_0. Going from $t = 0$ to $t = 1$ and onward reverses the collection of pictures.

In fact, *all* non-empty cross sections of the sphere are circles of varying radii. The sphere is the only surface in $\mathbb{R}^3$ with this property. Reviewing our work above, note that the "spun" surfaces also have circular cross-sections, parallel to the xy plane. Sections taken parallel to the other coordinate planes are more interesting.

Exercise 5.1. Let H be the hyperboloid of one sheet. Sketch pictures of all cross-sections of H, taken parallel to the xz plane. Note that the xz plane is itself a section.

Exercise 5.2. The set $C : x^2 + y^2 = z^2$ is called the *cone*. Check that examples of all of the conic sections (ie, quadratic curves in two dimensions) may be obtained as cross sections of the cone. This is *why* they are called conic sections!

Exercise 5.3. Prove that every ellipsoid (including ones not obtained by spinning) have a circular cross section.

6 Parametrized curves and surfaces

Explicitly defined sets are given by *functions*. As we have seen, any line in $\mathbb{R}^3$ has the form $L(t) = p + tv$. Here p is a point in space and v is a vector. You can think of the function $L(t)$ as telling you the position of