1 Parametrized curves and surfaces

Explicitly defined sets are given by *functions*. As we have seen, any line in $\mathbb{R}^3$ has the form $L(t) = p + tv$. Here p is a point in space and v is a vector. You can think of the function $L(t)$ as telling you how the position of of a point changes with time. Now, if $p = (p_1, p_2, p_3)$ and $v = (v_1, v_2, v_3)$ then we find that $L(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$. That is, a line in three-space is given by three simultaneous linear functions.

Generally, *space curve* is given by a triple of one-variable equations. For example, the *helix* is $H(t) = (\cos(t), \sin(t), t)$. Here the variable t is often called the *parameter* of the curve. Again, you can think of a space curve as recording the position, $P(t)$, of a moving particle. The set of all positions of the particle is sometimes called the *image*, or *track* of the function P. Note that the image of a space curve is onchanged if we *reparametrize*: that is, suppose that $C(t)$ is a space curve, and $f(t)$ is a one variable function. Then $D(t) = C(f(t))$ is another space curve. Often you cannot tell the curves C and D apart by looking at their images. Instead of effecting *where* the particle goes, the reparametrization effects *how fast* the particle is going.

Exercise 1.1. Suppose that a particle moves through space and has position $H(t) = (\cos(t), \sin(t), t)$ at time t. How "fast" is the particle going? At time t, in what direction is the particle moving? (We will discover systematic ways of answering these questions, below.)

Exercise 1.2. All of the above discussion also holds in two dimension: A *plane curve* is given by a pair of one-variable equations. For example, can you sketch the image of the plane curve $T(t) = (\cos(3t), \sin(2t))$?

The simplest plane and space curves arise as *graphs*: given a one-variable function $f(t)$ we can form the plane curve $F(t) = (t, f(t))$. The image of F is the graph of f. Likewise, if we have a plane curve, say $g(t) = (h(t), k(t))$, then the graph of g is the image of the space curve $G(t) = (h(t), k(t), t)$. (Where to put the t is not really standard – if you use it as the x coordinate instead of the z coordinate the graph changes, but only by a reflection.)

2 Examples

Here is another pretty space curve: $S(t) = \left(\frac{\cos(t)}{\sqrt{1+t^2}}, \frac{\sin(t)}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}\right)$. This pretty clearly has something to do with the helix, but what?

Exercise 2.1. Sketch the image of S either by hand, or by using Maple. If a particle p moves according to the space curve $S(t)$ then how "fast" is p moving as $t \to \pm\infty$? How does the motion of p along S compare to the motion of a particle along the helix H?

The book gives additional examples which it calls *toroidal curves*. For example, suppose that a particle p follows

$$C(t) = (\cos(t), \sin(t), 0).$$

This is just motion along a circle in the xy plane. We can *perturb* this motion by, at time t, pushing the particle p just a little bit in some other direction. For example, suppose that ϵ is a very small number. Let

$$D(t) = \epsilon \left(\cos\left(\frac{3}{2}t\right)\cos(t), \cos\left(\frac{3}{2}t\right)\sin(t), \sin\left(\frac{3}{2}t\right) \right).$$

We can now form the new space curve $E(t) = C(t) + D(t)$ which looks *almost* like $C(t)$, but pushed slightly off.

Exercise 2.2. Sketch $E(t)$ with $\epsilon = 1/4$ and with t in radians. How does the image change if you change the $\frac{3}{2}$ to another number, say $\frac{5}{3}$ or $\frac{8}{5}$? What if you take the coefficient to be irrational?

This sort of thing has serious applications – for example we are not too far from giving a toy model of the Sun-Earth-Moon system this way.

Exercise 2.3. Here is yet another example, the *twisted cubic*: $T(t) = (t, t^2, t^3)$. This curve (and its cousins) show up in *algebraic geometry*. Prove that if a, b, c, and d are distinct real numbers then the points $T(a)$, $T(b)$, $T(c)$, and $T(d)$ do *not* lie in a single plane in $\mathbb{R}^3$. (Hint: solve the corresponding problem for the parabola in $\mathbb{R}^2$, first.)

3 Tangents

Suppose that $p(t)$ records the position, at time t, of a particle p moving through $\mathbb{R}^n$. There are two immediate questions of interest at time t: how fast is p moving and in what direction?

You have already seen the answer to this question if $n = 1$: ie if p moves along a straight line. The magnitude of $p'(t)$ measures the speed and the sign of $p'(t)$ tells us if p is moving in the positive or negative x-direction.

In n-space these questions have similar answers. Choose h a small real number. Then the length of the vector $p(t + h) - p(t)$, divided by h, tells us *approximately* how fast p is moving at time t. Similarly, the direction of $p(t+h)-p(t)$ tell us approximately the direction in which p moves at time t. So both pieces of information are recorded by the rescaled vector:

$$\frac{p(t + h) - p(t)}{h}.$$

However this is only an approximation. For well behaved motions $p(t)$ the limit

$$p'(t) = \lim_{h \to 0} \frac{p(t+h) - p(t)}{h}$$

exists. Now, just as a limit of points is again a point, so a limit of vectors is again a vector. We dub $p'(t)$ the *tangent vector* to the path of p at time t.

An important special case occurs when $|p'(t)|$ is constant. In this case we say that $p(t)$ is a *constant speed* path. Of course, if $|p'|$ is identically zero, then the path is a *constant path*: the particle is not moving.

As a notational short cut, we will often pretend that $p(t)$ is not a path of points but rather is a path of vectors, all based at the origin. This should cause no confusion.

If $p(t) = (x(t), y(t), z(t))$ is a presentation of $p(t)$ in coordinates then we find that $p'(t) = (x'(t), y'(t), z'(t))$. As usual, explicit coordinates are to be avoided, if possible. As an example we offer the following lemma:

Lemma 3.1. *If $p(t)$ and $q(t)$ are space curves then $(p \cdot q)' = p' \cdot q + p \cdot q'$.*

Proof. Of course it is possible to prove this using coordinates. Let's avoid the extra work as follows. Notice that $p(t+h) \sim p(t) + hp'(t)$ where $\sim$ means "about equal". A similar "equality" holds for q. To compute $(p \cdot q)'$ we must consider

$$\frac{p(t+h) \cdot q(t+h) - p(t) \cdot q(t)}{h}.$$

This is about equal to

$$\frac{(p(t) + hp'(t)) \cdot (q(t) + hq'(t)) - p(t) \cdot q(t)}{h}.$$

By linearity of the dot product ($(v + w) \cdot u = v \cdot u + w \cdot u$) the above equals

$$p(t) \cdot q'(t) + p'(t) \cdot q(t) + hp'(t) \cdot q'(t).$$

As $h \to 0$ the third term drops out and we have the desired derivative. $\square$

As a corollary we note the important fact: if $|p|$ is constant then so is $|p|^2 = p \cdot p$ and taking a derivative shows that $p(t)$ is orthogonal to $p'(t)$.

A similar formula to Lemma 3.1 holds for the cross product:

Lemma 3.2. *If $p(t)$ and $q(t)$ are curves in $\mathbb{R}^3$ then $(p \times q)' = p' \times q + p \times q'$.* $\square$

We pause to note that $p \times p$ is always zero. Differentiation yields the banal fact that $p \times p' + p' \times = 0$, which already follows from the definition of the cross product.

4 Arclength

The *unit normal* to a space curve $p(t)$ is the unit vector in direction $p'(t)$. The unit vector is denoted by $T(t) = \frac{p'(t)}{|p'(t)|}$. The line given by T is the *tangent line* to p at time t. We say that p is a *unit speed* curve if no rescaling is necessary: if $|p'(t)| = 1$ for all time.

If $p(t)$ is *not* a unit speed curve it is sometimes necessary to reparametrize to *make* it unit speed. All of the formulae in the next section are simpler for unit speed curves.

Add more here...

5 Curvature

For the moment we assume that $p(s)$ is a unit speed curve. It follows that $\frac{dp}{ds} = T$ and that $T \cdot \frac{dT}{ds} = 0$. We define the *curvature* of p to be

$$\kappa(s) = \left|\frac{dT}{ds}\right|.$$

This is a sophisticated notion, involving as it does *second* derivatives of the motion of p. The tangent tells us the direction of motion of p. The curvature tells us how rapidly that direction is changing.

Exercise 5.1. Check that both the circle $C(t) = (\cos(t), \sin(t))$ and the helix $H(t) = (\cos(t), \sin(t), t)$ have constant speed and curvature. Check that the unit speed curve p has curvature zero iff p describes a line.

Exercise 5.2. Of course, all of these computations also work for plane curves. Make a guess to the curvature of the ellipse $E(t) = (a\cos(t), b\sin(t))$. Compute the curvature. Careful: the given parameterization is not unit speed, nor is it possible to reparametrize to obtain a unit speed curve with the same image. (At least, *I* don't know how!)

We have already seen that dT/ds is perpendicular to T. It's length is given by $\kappa(s) = |dT/ds|$. So we define the *normal* vector

$$N(s) = \frac{dT/ds}{|dT/ds|}.$$

Together N and T span the *normal plane* to p at time s. As both are unit length so is the *binormal* $B = T \times N$, which is perpendicular to the normal plane.

Again, as B has unit length deduce that dB/ds is perpendicular to B. Looking at the derivative (with respect to s) of $B \cdot T = 0$ it is possible to deduce that dB/ds is also perpendicular to T. (This is a hint for one of your homework problems.) It follows that dB/ds and N point along the same line. So we define the *torsion* of the curve p by the formula:

$$dB/ds = -\tau(s)N.$$

The negative sign is there by convention.

Exercise 5.3. We have formulae $dT/ds = \kappa N$ and $dB/ds = -\tau N$. Give a similar formula for dN/ds.