1 One-variable

Recall the definition of the one-variable functions. It is defined to be

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

wherever defined. This measures the *rate of change* of f in the direction of x.

Exercise 1.1. Can you express $\lim_{h\to 0} \frac{f(t+2h)-f(t)}{h}$ in terms of $f'(t)$? What about $\lim_{h\to 0} \frac{f(t+ah)-f(t)}{h}$? (Note that t and a are both constant in the above limit. Only h is changing.)

2 Multivariable functions

As a first example: a weather map takes points on the surface of the earth (2-dimensional, so requires two-variables) and gives back a single number. An *isothermal lines* is a curve on the earth of all the places with the same temperature. This is a special case of *level curve*.

There is a close connection between implicit plane curves (such as $C : x^2 + y^2 = 1$), level curves of two-variable functions ($f(x, y) = x^2 + y^2$), and implicit surfaces ($P : z = x^2 + y^2$). We have seen the sets $C \subset \mathbb{R}^2$ and $P \subset \mathbb{R}^3$ but not the function $f(x, y)$. Notice that that the implicit surface P is the *graph* of $f(x, y)$.

The weather map example extends to three dimensions – think of a function which measures the temperature in the room you are sitting in. The function takes a point and returns the temperature of that point. All of the points near the ceiling are a bit warmer than the points further down, because heat rises. The points close to your body are also a bit hotter, because humans radiate. Of course, instead of isothermal curves, in dimension three we have isothermal *surfaces*, wrapping us like a blanket...

Exercise 2.1. Perhaps it is not yet time to discuss general functions $f : \mathbb{R}^n \to \mathbb{R}^m$ taking multiple variables to multiple variables. But if we did then the graph of f lives in $\mathbb{R}^{n+m}$. These graphs can be understood using cross-sections, just as we did with surface in $\mathbb{R}^3$. For example, we have already discussed in class, in some sense, the $w = c$ cross-sections of $w = f(x, y, z) = x^2 + y^2 - z^2$.

3 Derivatives

To fix ideas let's look at a function $f(x, y) = x^2 + y^3$. Suppose that we are interested in the point $P = (1, 1)$. In one-variable there is only one derivative (up to scale!) and it is obtained by adding h to the x variable. This is because in dimension one there is only one direction to move in. We now have two independent directions. Let's choose one, say the vector $V = (2, -1)$. Then we can plug in $P + hV$ into f to vary the function. We define the *directional derivative*:

$$f_V(P) = \lim_{h \to 0} \frac{f(P + hV) - f(P)}{h}.$$

To get a "number" we can decend to coordinates: compute $f(1,1) = 2$ and $f(P + tV) = (1+2h)^2 + (1-h)^3$. So $f(P+hV) - f(P) = 1 + 4h + 4h^2 + 1 - 3h + 3h^2 - h^3 - 2 = h + 7h^2 - h^3$. Divide by h and take $h \to 0$ (in that order!) to get $f_V(P) = 1$. So the rate of change of f at $(1,1)$ in the direction $(2,-1)$ is 1. Another way to think of this: The surface $z = x^2 + y^3$ has a tangent line with slope one lying over the line $L(t) = P + tV = (1,1) + t(2,-1)$ in the xy plane. To find the formula of the tangent line, we just need to fill in the third coordinate: $T(t) = (1,1,2) + t(2,-1,1)$.

In general, if $P = (x,y)$ and $V = (v,w)$ then then the "tangent line to the graph of f at the point $(x, y, f(x,y))$ in the direction V" is the line $T(t) = (P, f(P)) + t(V, f_V(P)) = (x, y, f(P)) + t(v, w, f_V(P))$. Don't get confused here – there is only one variable, t!

Exercise 3.1. Before reading on: suppose that $f(x,y) = x^2 + y^3$. At the point $P = (1,1)$ compute the tangent lines T_θ in the directions $V_\theta = (\cos(\theta), \sin(\theta))$ as θ varies between 0 and 2π. What space curve does $T_\theta(1)$ describe?

There are a few directional derivatives which deserve special names: we take f_x to be the direction derivative in the direction $\mathbf{i}$, f_y in the direction $\mathbf{j}$, and f_z in the direction $\mathbf{k}$. For example, if $f(x,y,z) = x^2 + y^3 + xz$ then $f_x(x,y,z) = 2x + z$, $f_y(x,y,z) = 3y$, and $f_z(x,y,z) = x$. In general, given $g(x,y,z)$ you can compute g_x by holding y and z fixed and differentating with respect to x. (Again, this is just the direction derivative in the $\mathbf{i}$ direction.)

We put these special derivatives together in a package:

$$\nabla f = (f_x, f_y, f_z)$$

or, in dimension two:

$$\nabla f = (f_x, f_y).$$

This is called the *gradient* of f. (The ∇ symbol is called nabla, for some reason?) The function $f_x(x,y,z)$ is also called the *partial derivative* of f in the direction of x.

4 Tangent planes

We have two ways of presenting sets: implicitly, as the level set of a function, and explicitly, with a parameterization. (Graphs of functions are either, depending on how they are given. If we write $S : z = x^2 + y^3$ the graph is implicit. If we write $S : (x, y, x^2 + y^3)$ it is explicit. As we have seen, it is not always so easy to rewrite an implicit set as an explicit one!)

Today we will find tangents to implicit sets, ie to level curves. Let $f(x, y) = x^2 + y^3$, say. As above take $P = (1, 1)$. Then P lies on the level curve $C : x^2 + y^3 = 1$ of the function f. We wish to find the tangent to C at P. But you will remember that this was covered in one-variable calculus using *implicit differentiation*: compute $\frac{d}{dx}$ of both sides and solve for $\frac{dy}{dx}$. This gives the slope of the tangent line and so gives the tangent line.

Exercise 4.1. Do this now.

Now lets do this the multivariable way: We have $f(x, y) = x^2 + y^3$. So $f_x(x, y) = 2x$, $f_y(x, y) = 3y^2$, and so $\nabla f = (2x, 3y^2)$. At the point P we find $\nabla_P f = (2, 3)$. We *define* the tangent line to be $L : \nabla_P f \cdot V_P = 0$. That is, the tangent line is the set of all vectors based at P which are orthogonal to the gradient. (Alternatively, T may be thought of as the set of points Q satisfying $\nabla_P f \cdot (Q - P) = 0$.) Does this agree with the line you computed in the exercise above?

Generally, if $g\colon \mathbb{R}^n \to \mathbb{R}$ then $g = c$ describes a level curve ($n = 2$), surface ($n = 3$), space ($n = 4$), etc. Call it L_c. Let tangent line, plane, space etc to L_c at P be T. Then T is given by:

$$T : \nabla_P g \cdot V_P = 0.$$

Now, just as graphs of one-variable functions have their tangent planes, so do the graphs of two-variable functions have tangent planes. (Generally, a map $f\colon \mathbb{R}^n \to \mathbb{R}$ has a tangent n-space at every point of its graph. Here we are tacitly assuming that f has directional derivatives in all directions and that these are continuous.)

As we shall see, the tangent plane to a graph at P is *also* equal to the union of all tangent *lines* to the graph at P. However, to prove this we will need to investigate the *chain rule*.