Problem 5.1. Let D be the region in $\mathbb{R}^3$ bounded by the ellipsoid $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Compute the volume of D. (You may use without proof the fact that the unit ball, bounded by $S^2 : x^2 + y^2 + z^2 = 1$, has volume $\frac{4}{3}\pi$.)

Problem 5.2. Consider the transformation $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Let D be the rectangle with vertices $(0, 0)$, $(1, 0)$, $(0, 2\pi)$, and $(1, 2\pi)$. As done in class:

1. Find the images of the sides of D.
2. Find $T(D)$, the image of D.
3. Find the image of all of the horizontal and vertical lines in D.
4. What would happen if we increased the height of D from 2π to 4π?
5. Compute the total derivative of T.
6. Discuss how the area of a tiny subrectangle of D changes when you apply T. How does the change in area depend on r and θ?

Now derive the change of variables formula for polar coordinates:

$$\iint_{T(C)} f(x, y)\, dA = \iint_C f(T(r, \theta))\, r\, dr\, d\theta.$$

Here C is a rectangle in r, θ coordinates.

Problem 5.3. (See also Problem 36, page 1009 of Stewart.) The goal of this problem is to compute $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. This is defined to be the limit $\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} dx$. We proceed as follows:

1. Square I and change the variable of integration of the second copy of I to be y instead of x.
2. Rewrite I^2 as a multivariable integral. Give the definition of this indefinite multivariable integral. (Hint: integrate over squares with sidelength $2a$, centered at the origin. Take a limit.) Draw a picture.
3. Change to polar coordinates. *Briefly* discuss why you can take the limit of the integral over disks of radius a, instead of squares. (That is; argue why the difference of the two integrals goes to zero as $a \to \infty$.) Draw a picture.
4. Find the value of the resulting improper integral by computing the polar integral over a disk of radius a and taking a limit.
5. Take a square root to find the value of I.