Please *write* this week's workshop with at most one other person. Be sure to acknowledge any and all help you receive with your write-up. As usual, you can *think* with other people but you should not *write* with more than one other person. Please write your answer in complete English sentences. Helpful figures are always welcome.

Todays workshop seeks to explain all of the "derivatives" we have seen (or will see) in class. Recall that ∇ is the "del" operator. We have $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ in dimension two and $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ in dimension three. Now let us list all the derivatives in dimension two:

- If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a scalar function then we defined $\nabla f = \langle f_x, f_y \rangle$, the *gradient* of f.
- If $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, say $F = \langle P, Q \rangle$, then we defined $\nabla \times F = Q_x - P_y$, the *curl* of F. This is sometimes written curl F.

Here are all of the derivatives in dimension three:

- As above, if $f\colon \mathbb{R}^3 \to \mathbb{R}$ then $\nabla f = \langle f_x, f_y, f_z \rangle$ is the gradient of f.
- Similar to the above, if $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, say $F = \langle P, Q, R \rangle$, then we define
$$\nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$
This is also called the curl of F and sometimes written curl F.
- Also, if $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then we define $\nabla \cdot F = P_x + Q_y + R_z$, the *divergence* of F. This is sometimes written div F.

Problem 8.1. Check that, in dimension two, $\nabla \times (\nabla f) = 0$ for any function f. That is, the curl of the gradient is always zero. (If you have trouble doing this, check a few examples like $f(x, y) = x^2 + y^2$ or $g(x, y) = e^x \cos(y)$. Clairaut's Theorem on page 916 may be helpful.)

Problem 8.2. Check that this also holds in dimension three $\nabla \times (\nabla f) = 0$. (Again, do a few examples first: like $f(x, y, z) = xyz$ or $g(x, y, z) = ye^{x^2+z}$.) The fundamental theorem for line integrals explains why this works – gradient vector fields are always conservative.

Problem 8.3. Also in dimension three, check that
$$\nabla \cdot (\nabla \times F) = 0.$$

The above three exercises can be summarized by saying "taking two derivatives yields zero." This is very similar to the fact, observed in class, that $\partial\partial D = \emptyset$: for any domain D, the boundary of the boundary of D is empty.

Problem 8.4. Recall that the one variable product rule says $(fg)' = f'g + fg'$. Similar rules hold in higher dimensions. For example: suppose that f and g are both functions on $\mathbb{R}^3$. Find a formula for the gradient of the product fg in terms of ∇f and ∇g.

Problem 8.5. If f is a function and F is a vector field, both in the same dimension, then we can define a new vector field by scaling: $G = fF$. Derive formulas for $\nabla \times G$ (in dimensions two and three) and for $\nabla \cdot G$ in terms of f, F, and their derivatives.

Problem 8.6. (Hard.) Here is a final product rule problem: suppose that F and G are both vector fields. Find expressions for the derivatives of $F \cdot G$ and $F \times G$ in dimensions two and three. (Compare to problem 20 on page 1136 of the book.) An important special case is the gradient of $|F|$.