Please *write* this week's workshop with at most one other person. Be sure to acknowledge any and all help you receive with your write-up. As usual, you can *think* with other people but you should not *write* with more than one other person. Please write your answer in complete English sentences. Helpful figures are always welcome.

Today's workshop will concentrate on a single object: the solid torus standardly embedded in $\mathbb{R}^3$. Pick two numbers $a > b > 0$. Form in the $xz$ plane the circle of radius $b$, centered at the point $(a, 0, 0)$. Rotate this about the $z$ axis to obtain a solid of revolution which we will label by $T$ or $T(a, b)$. The boundary, $\partial T$, is two-dimensional and is called a "torus." Note that $T$ is called a "solid torus."

**Problem 10.1.** Draw the cross sections of $T$ and $\partial T$ which are parallel to the coordinate planes. What is the difference between those of $T$ and those of $\partial T$?

Give a sketch of $T$ in the $xyz$ space, with the coordinate axes in the standard position.

**Problem 10.2.** Using one variable techniques (ie shells) find the volume $V(a, b)$ of $T(a, b)$. (Hint: remember to look for symmetry in the integrand, such as even or odd functions.)

**Problem 10.3.** Using one variable techniques (ie strips) find the area $S(a, b)$ of $\partial T(a, b)$. Once you have computed $V(a, b)$ and $S(a, b)$ try to rewrite them in a geometrically meaningful way.

Now that we've recalled some of the difficulties of one-variable calculus, let's find out if our fancy three dimensional techniques are any easier.

**Problem 10.4.** Let $Q$ be the solid in $uvw$ space bounded by the cylinder $u^2 + w^2 = 1$ and the pair of planes $v = 0$ and $v = 2\pi$. Put another way, $Q = \{(u, v, w) \mid u^2 + w^2 \leq 1, v \in [0, 2\pi]\}$. Sketch $Q$ in $uvw$ space, with the coordinate axes in the standard position. Find the volume of $Q$. Find the surface area of $Q$.

Now give a nice transformation $\mathbf{r}\colon Q \to T$ from $uvw$ space to $xyz$ space which throws $Q$ onto $T$. The point $(0, 0, 0)$ should be sent to $(a, 0, 0)$ and the $v$-axis should be sent to the "core circle" of $T$. Also, $\mathbf{r}_w$ should be a constant multiple of the vector $\mathbf{k}$. Where are planes, parallel to the $uv$, $vw$, and $wu$-coordinate planes, sent by $\mathbf{r}$?

**Problem 10.5.** Writing the parameterization $\mathbf{r}$ as $\mathbf{r} = (x, y, z)$ (where each of $x$, $y$, and $z$ are functions of the variables $u$, $v$, and $w$) compute $dx$, $dy$, and $dz$ in terms of $du$, $dv$, and $dw$. Pause to admire the total derivative.

**Problem 10.6.** Compute the integral $\iiint_T dV = \iiint_T dx\, dy\, dz$ using the map $\mathbf{r}$.

**Problem 10.7.** Let us think quite generally for a moment: Suppose that $F$ is any nice surface in $\mathbb{R}^3$. Suppose that $\mathbf{r}\colon D \to F$ is a map from the $\phi\theta$ plane to $xyz$ space throwing a region $D$ onto the surface $F$. Then the book claims that the surface area of $F$ is:

$$\iint_F dF = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi\, d\theta.$$

Prove that the latter integral is identical to

$$\iint_F \sqrt{(dy\,dz)^2 + (dz\,dx)^2 + (dx\,dy)^2}.$$

(We take as a convention that $\sqrt{\omega^2} = \omega$ for any form $\omega$, as we did when computing arclength.)

**Problem 10.8.** Choose a nice map $\mathbf{s}(\phi, \theta)$ from the $\phi\theta$ plane to the $uvw$ space which throws the square $[0, 2\pi]\times[0, 2\pi]$ onto the cylindrical part of $\partial Q$. (Hint: take $v(\phi, \theta) = \phi$.) Compute a totally explicit expression for $\mathbf{R}(\phi, \theta) = \mathbf{r}(\mathbf{s}(\phi, \theta))$. Use this to again find $S(a, b)$, the area of $\partial T$, using one of the formulae from Problem 10.7.

**Problem 10.9.** What is the relationship between $V(a, b)$ and $S(a, b)$? Is one, as perhaps claimed in class, the "derivative" of the other?