Here we record several useful facts about forms. To begin with we have the (somewhat loose) definition:

Definition 11.1. A *k-form* in n-space is something you integrate over a k dimensional domain.

Here are all possible examples in dimension one, two, and three:

dim	0-form	1-form	2-form	3-form
one	P	P dx	none	none
two	P	P dx + Q dy	P dxdy	none
three	P	P dx+Q dy+R dz	P dydz+Q dzdx+R dxdy	P dxdydz

Here P, Q, and R are scalar functions. All facts about forms in dimensions one and two can be recovered from the corresponding fact in dimension three. So we restrict our attention to $\mathbb{R}^3$ from now on.

We now can discuss the transformations between functions and vector fields on the one hand and forms on the other hand. Suppose that P is a function on three-space. If we want to evaluate P at a point we think of P as a zero-form. If we want to integrate P over a volume we think of the three-form $P\,dxdydz$. Now fix a vector field $F = \langle P, Q, R\rangle$. If we want to integrate F over a curve we think of F as a one-form $P\,dx + Q\,dy + R\,dz$. If we want to integrate F over a surface we instead think of F as a two-form $P\,dydz + Q\,dzdx + R\,dxdy$.

Given any form ω we can take a derivative $d\omega$. Here is a list of all possible derivatives.

dim	ω	$d\omega$
zero	P	$P_x\,dx + P_y\,dy + P_z\,dz$
one	$P\,dx + Q\,dy + R\,dz$	$(R_y - Q_z)\,dydz + (P_z - R_x)\,dzdx + (Q_x - P_y)\,dxdy$
two	$P\,dydz + Q\,dzdx + R\,dxdy$	$(P_x + Q_y + R_z)\,dxdydz$
three	$P\,dxdydz$	0

Of course you do not need to memorize the list of derivatives. Instead just remember:

- the derivative of P is $dP = P_x\,dx + P_y\,dy + P_z\,dz$, just like the gradient,
- the product rule $d(P\omega) = dP\,\omega + P\,d\omega$,
- the orientation rule $dydx = -dxdxy$, and
- and the fact that $dxdx = 0$.

From these rules and Clairaut's Theorem you can deduce for any form ω the double derivative $dd\omega$ vanishes. That is, for any form ω, we have $dd\omega = 0$. As a bit of notation, for any form ω, if $d\omega = 0$ we say ω is *closed*. On the other hand if there is a form τ so that $\omega = d\tau$ we say ω is *exact*. As all double derivatives vanish, we find that *all exact forms are closed.*

Exercise 11.2. Notice that if you transform a function into a zero-form, take the derivative, and transform the resulting one-form back to a vector field then you obtain the gradient of the original function. If you begin by transforming a vector field into a one-form then the eventual result will be the curl. If you begin by turning a vector field into a two-form then the result will be the divergence.

It follows from the above exercise that the vector field of workshop problem 9.4 gives a one-form which is closed but *not* exact. It is straight-forward to rewrite the fundamental theorem of line integrals in terms of forms:

Theorem 11.3. *Suppose that ω is a one-form defined on a simply-connected domain $D \subset \mathbb{R}^3$. Then the following properties are equivalent:*

- *ω is* conservative*: $\int_C \omega = 0$ for any closed curve C in the domain D.*
- *ω is* independent of path*: the value of $\int_C \omega$ depends only on the endpoints of the path C.*
- *ω is exact: there is a zero-form P so that $\omega = dP$.*
- *ω is closed: $d\omega = 0$.*

Stokes' Theorem ($\int_{\partial B} \omega = \int_B d\omega$) is used only to prove that the last property implies the first. Thus, the first three properties remain equivalent (and imply the fourth) in *any* domain.

Exercise 11.4. I invite the reader to generalize the fundamental theorem for line integrals to surfaces.