Math 311: Section 3.

Handout: Axioms for $\mathbb{R}$.

We won't give a construction of $\mathbb{R}$ in this class. If you are interested in a precise construction, you could read up on the theory of *Dedekind cuts* either in Abbot's book (Section 8.4) or in Rudin's.

Instead, we will be satisfied with the following list of axioms. We are given a set $\mathbb{R}$ with two binary operations $(x, y) \overset{+}{\mapsto} x + y$ and $(x, y) \overset{\cdot}{\mapsto} xy$ satisfying the following properties:

The field axioms:

- Right identities: There are elements $0 \neq 1$ in $\mathbb{R}$ so that for all $x \in \mathbb{R}$ we have $x + 0 = x$ and $x \cdot 1 = x$.
- Right inverses: For all $x \in \mathbb{R} \setminus \{0\}$ there are elements $-x, 1/x \in \mathbb{R}$ so that we have $x + (-x) = 0$ and $x \cdot (1/x) = 1$.
- Commutativity: For all $x, y \in \mathbb{R}$ we have $x + y = y + x$ and $xy = yx$.
- Associativity: For all $x, y, z \in \mathbb{R}$ we have $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.
- Right distributivity: For all $x, y, z \in \mathbb{R}$ we have $(x + y)z = xz + yz$.

Moreover, $\mathbb{R}$ is an *ordered* field: there is a binary relation $<$ with the following properties:

The ordered field axioms:

- Trichotomy: For all $x, y \in \mathbb{R}$ we have exactly one of the following: $x = y$, $x < y$, or $y < x$.
- Transitivity: For all $x, y, z \in \mathbb{R}$ if $x < y$ and $y < z$ then $x < z$.
- Order distributivity: For all $x, y, z \in \mathbb{R}$ if $x < y$ then $x + z < y + z$.
- Positivity: For all $x, y \in \mathbb{R}$ if $0 < x$ and $0 < y$ then $0 < xy$.

As a bit of notation, for all $x, y \in \mathbb{R}$ we write $x \leq y$ whenever either $x = y$ or $x < y$. Finally $\mathbb{R}$ is *complete*:

The axiom of completeness: Every non-empty subset of $\mathbb{R}$ admitting an upper bound has a supremum.

Recall the definitions: Suppose that $A \subset \mathbb{R}$ is non-empty. We say $x \in \mathbb{R}$ is an *upper bound* for A if for all $a \in A$ we have $a \leq x$. We say $x \in \mathbb{R}$ is a *supremum* for A, and write $x = \sup A$, if x is an upper bound for A and for any other upper bound y for A we have $x \leq y$.