Math 311: Section 3.

Handout: When is $\sqrt{n}$ irrational?

Here is one answer to Problem 0.4 from Workshop 1, adapted from a student's paper. Recall that $n \in \mathbb{N}$ is a *perfect square* if there is an $m \in \mathbb{N}$ so that $n = m^2$.

Theorem 0.1. *If $n \in \mathbb{N}$ then either n is a perfect square or $\sqrt{n}$ is irrational.*

Proof. Fix $n \in \mathbb{N}$. Note that $\sqrt{n} \geq 1$.

Exercise 0.2. Can you prove if $n \in \mathbb{N}$ then $\sqrt{n} \geq 1$, directly from the ordered field axioms? To be precise: suppose that a is a positive real number. Prove that $a \geq 1$ if and only if $a^2 \geq 1$.

Suppose that $\sqrt{n}$ is rational. Then there are many ways to write $\sqrt{n}$ as a fraction. Let $B = \{b \in \mathbb{N} \mid \exists a \in \mathbb{N} \text{ with } \sqrt{n} = a/b\}$. The set B is nonempty because $\sqrt{n}$ is rational and positive. By the well-ordering principle for $\mathbb{N}$ there is a smallest element $q \in B$.

We may now write $\sqrt{n} = p/q$. It follows that $nq^2 = p^2$. As $\sqrt{n} \geq 1$ we also have that $p \geq q$. Using long division, we may write $p = qm + r$ where $m \in \mathbb{N}$ and r is an integer, $0 \leq r < q$.

Exercise 0.3. Long division is an *algorithm* and, strictly speaking, its correctness requires a proof. Write down the algorithm and give a proof (necessarily by induction) that the algorithm always gives correct output. As a hint: the input to the algorithm is two natural numbers p and q. What should you induct on?

Suppose now that $r = 0$. Then $\sqrt{n} = \frac{qm+r}{q} = \frac{qm}{q} = m$. It follows that $q = 1$ and that $n = m^2$. So n is a perfect square.

Suppose instead that $r > 0$. Since $q > r$ it follows that $q > 1$. Recall that $nq^2 = p^2$. Subtract mpq from both sides to find $nq^2 - mpq = p^2 - mpq$. Factor to find $q(nq - mp) = p(p - mq) = pr$. As $r, q \neq 0$ we can cross-divide to find $\sqrt{n} = p/q = (nq - mp)/r$.

Exercise 0.4. Show that $nq - mp$ is positive.

So we have found another way to write $\sqrt{n}$ as a fraction, with positive numerator, but with a smaller denominator. This contradicts the fact that q is the smallest element of B.

It follows that $\sqrt{n}$ is either a perfect square or is irrational. $\square$

Exercise 0.5. Why does this proof work? How can we justify the "magic step" where we subtracted mpq from both sides?