Math 311: Section 3.

Workshop 3: Exciting series!

Recall that the *series* $\sum_{i=1}^{\infty} x_i$ corresponds exactly to the sequence $\{p_n\}_{n=1}^{\infty}$ where $p_n = \sum_{i=1}^{n} x_i$ is the n^{th} *partial sum*. We say the series $\sum_{i=1}^{\infty} x_i$ *converges* exactly when $\{p_n\}$ does and we write $\sum_{i=1}^{\infty} x_i = x$ if $p_n \to x$.

Proving that a series converges is often a delicate affair. Example 2.4.4 in the book gives a very good model, if you get stuck.

When a series *does* converge actually finding the limit can be even more difficult: often this requires an understanding of the algebraic nature of the given series. In the problems that follow, you are only asked to prove that the series converges — however you should also experiment with finding the limit, and trying to prove that your guess is correct.

Here is a bit of notation before we start. Recall that $n!$ denotes the product of the first n natural natural numbers: $n! = 1\cdot 2\cdot 3\dots(n-1)\cdot n$. By convention we take $0! = 1$. We also define the *double factorial*: $(2n+1)!! = 1\cdot 3\cdot 5\dots(2n-1)\cdot(2n+1)$. By convention we take $(-1)!! = 1$. When dealing with series it often helps to think in terms of *estimates*. For example, can you prove that $n! \geq n^2$ for all $n > 3$? How about showing that $n! \geq 2^{n-1}$ for all $n > 1$? Can you prove that $(2n-1)!! < n!\cdot 2^n$ for all n? Or that $n! < \left(\frac{n+1}{2}\right)^n$ for all n?

Problem 3.1. Give a complete proof that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

converges.

Problem 3.2. Show that the series (due to Euler)

$$2\left(\sum_{n=2}^{\infty} \frac{n!}{(2n-1)!!}\right) = 2\left(\frac{2!}{3!!} + \frac{3!}{5!!} + \frac{4!}{7!!} + \frac{5!}{9!!} + \dots\right)$$

converges.

Problem 3.3. Here is another nice series, essentially due to Newton:

$$2\left(1 - \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!\cdot 2^{2n}}\right) = 2\left(1 - \frac{1}{2^2} - \frac{1!!}{2!\cdot 2^4} - \frac{3!!}{3!\cdot 2^6} - \frac{5!!}{4!\cdot 2^8} - \frac{7!!}{5!\cdot 2^{10}} - \dots\right)$$

Show that the series converges.