Math 311: Section 3.

Solutions for Workshop 6: Cantor set, Cantor set.

Solution 6.1. We sketch a proof of the fact that $C + C = \{x + y \mid x, y \in C\} = [0, 2]$, where C is the Cantor set.

Proof. We begin by introducing a few pieces of notation. Suppose that A and B are subsets of $\mathbb{R}$. Suppose that $c \in \mathbb{R}$ is any real number. We define $A + B = \{a + b \mid a \in A, b \in B\}$. We define $c + A = \{c\} + A = \{c + a \mid a \in A\}$. We finally define $cA = \{ca \mid a \in A\}$. Note also that $A \subset A'$ and $B \subset B'$ implies that $A + B \subset A' + B'$. Finally, given sets A, B, C, D we have the "FOIL" identity: $(A \cup B) + (C \cup D) = (A + C) \cup (B + C) \cup (A + D) \cup (B + D)$

Now, let C be the Cantor set and let C_n be the n^{th} approximation to C. Note that $C \subset [0, 1]$. Thus, by the second to last remark of the previous paragraph we have $C + C \subset [0, 1] + [0, 1]$ and the latter is equal to $[0, 2]$. Thus $C + C \subset [0, 2]$. The opposite inclusion is more delicate and we turn to it now.

Fix $s \in [0, 2]$. We first must find, for all n, a pair of elements $x_n, y_n \subset C_n$ so that $x_n + y_n = s$.

Recall that $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on. It is easy to check, say by induction, that $C_{n+1} = \frac{1}{3}C_n \cup (\frac{2}{3} + \frac{1}{3}C_n)$.

Now, we have already shown that $C_0 + C_0 = [0, 1] + [0, 1] = [0, 2]$. Suppose now that $C_n + C_n = [0, 2]$. Note that

$$C_{n+1} + C_{n+1} = \left(\frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right)\right) + \left(\frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right)\right)$$

The sum of the first terms is $(\frac{1}{3}C_n + \frac{1}{3}C_n) = \frac{1}{3}[0, 2] = [0, \frac{2}{3}]$, by induction. Similarly the sum of the outer terms is $\frac{1}{3}C_n + (\frac{2}{3} + \frac{1}{3}C_n) = \frac{2}{3} + [0, \frac{2}{3}] = [\frac{2}{3}, \frac{4}{3}]$. The sum of the inner terms is $(\frac{2}{3} + \frac{1}{3}C_n) + \frac{1}{3}C_n = [\frac{2}{3}, \frac{4}{3}]$, giving the same set as the outer terms. Finally the sum of the last terms is $(\frac{2}{3} + \frac{1}{3}C_n) + (\frac{2}{3} + \frac{1}{3}C_n) = [\frac{4}{3}, 2]$.

The "FOIL" identity instructs us to take the union of these four sets. This gives $C_{n+1} + C_{n+1} = [0, 2]$, as desired. It follows that for all n there exists $x_n, y_n \in C_n$ so that $x_n + y_n = s$, the given value in $[0, 2]$.

Recall that $C_n \subset [0, 1]$ for all n. It follows that the sequence (x_n) is bounded. Thus by Bolzano-Weierstrass the sequence (x_n) admits a convergent subsequence (x_{n_l}). Suppose that (x_{n_l}) converges to $x \in [0, 1]$. Thus we have

$$\lim_{l \to \infty} y_{n_l} = \lim_{l \to \infty} (s - x_{n_l}) = s - x.$$

The last equality holds by Theorem 2.3.3. Thus the subsequence (y_{n_l}) also converges and converges to $y = s - x$.

All that is left to show is that $x, y \in C$, the Cantor set, and the proof will be complete. Recall that $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. So, for all $L \in \mathbb{N}$ and for all $l \geq L$ we have that x_{n_l} is contained in C_{n_L}. As C_{n_L} is closed (it is a finite union of closed intervals) we deduce that the limit x is contained in C_{n_L}. Finally, it follows that $x \in \bigcap_l C_{n_l}$. But this last intersection is exactly equal to the Cantor set. (Check this!) A similar argument shows that $y \in C$ and the proof is complete. $\square$

Solution 6.2. Fix a set of distinct real numbers, indexed by the natural numbers, $A = \{a_n\}_{n=1}^{\infty}$. Define

$$f_A(x) = \begin{cases} 0 & x \notin A \\ 1/n & x = a_n. \end{cases}$$

We will show that f_A is discontinuous at exactly the points of A. (We will use without proof the fact that the complement of A, A^c, is dense in $\mathbb{R}$.)

Proof. We begin by showing that f_A is discontinuous at the points of A. We use Corollary 4.3.3. Let (x_m) be a sequence in A^c converging to the point a_n in A. (This sequence exists because A^c is dense.) Then $\lim f(x_m) = 0 \neq 1/n = f(a_n)$ and thus f_A is discontinuous at all points of A.

We now must show that f_A is continuous at all points of A^c. Fix one point $b \in A^c$. So $f_A(b) = 0$. Fix now a value $\epsilon > 0$. We need to find a $\delta > 0$ small enough to guarantee that $|x - b| < \delta$ implies $|f_A(x) - f_A(b)| = |f_A(x)| < \epsilon$.

So choose some $N \in \mathbb{N}$ where $n > N$ implies that $1/n < \epsilon$. Let $\delta = \frac{1}{2} \min\{|a_j - b| \mid j = 1, 2, \ldots, N\}$. Then if $|x - b| < \delta$ there are two possibilities:

- $x \in A^c$ and $f_A(x) = 0 < \epsilon$ or
- $x = a_n$ for $n > N$ and so $f_A(a_n) = 1/n < \epsilon$.

In either case $|x - b| < \delta$ implies $|f_A(x)| < \epsilon$ and so f_A is continuous at b. We are done. □