Math 311: Section 3.

Workshop 9: An application of uniform convergence: finding a continuous function.

This workshop guides you through the book's discussion of the existence of a continuous nowhere differentable function (Section 5.4). Here we will only prove that the given function is continuous, leaving nondifferentability for another time.

Step 1. Define a function $h\colon \mathbb{R} \to \mathbb{R}$ by taking $h(x) = |x|$ if $|x| \leq 1$ and also taking $h(x+2) = h(x)$ for all $x \in \mathbb{R}$. That is, near 0 the function h looks like the absolute value function and h is 2-periodic. Graph h carefully. Prove that h is continuous.

Step 2. Set $h_n(x) = \frac{1}{2^n} h(2^n x)$. Graph h_1 and h_2 carefully. Give a qualitative description of of $h_n(x)$. What is the period of h_n? What is the magnitude $\sup\{h_n(x) \mid x \in \mathbb{R}\}$? Show that h_n is continuous for all n. (Hint: use theorems from the book.)

Step 3. For any $x \in \mathbb{R}$ and $m \in \mathbb{N}$ define $g_m(x) = \sum_{n=0}^{m-1} h_n(x)$. Show that for all $m \in \mathbb{N}$ the function g_m is continuous. (Hint: you do not need to prove this directly from the definition of continuity.)

Step 4. Fix $x \in \mathbb{R}$. Show that the sequence $(g_m(x))_{m=1}^{\infty}$ converges.

We *define* $G(x) = \lim_{m\to\infty} g_m(x)$. Deduce from the above that the function G is defined for all $x \in \mathbb{R}$. Deduce also that the functions (g_m) converge pointwise to G.

Step 5. Finally, show that the sequence (g_m) converges uniformly. (Hint: it is easier to understand what is going on if you use Theorem 6.2.5: the Cauchy criterion for uniform convergence. Also, the triangle inequality will be useful.) Deduce from the above and Theorem 6.2.6 that the function G is continuous.

Here are two further avenues of exploration. The first is clear: read and understand the book's discussion (page 146 to 147) proving that $G'(x)$ does not exist for any real number x. The second is more subtle, but easier: find a computer graphing package and use it to graph g_m for the first several values of m. How close to a graph of G can you get? What does the graph look like? What does it look like when you zoom in a bit?