Math 311: Section 3.

Workshop 10: On the Exponential function. (Or: "Why the Mean Value Theorem is still your friend.")

Do any two of the following problems. You must do at least one of Problem 10.1 or Problem 10.6.

Problem 10.1. Prove that the power series $E(x) = \sum \frac{x^n}{n!}$ converges uniformly on any interval of the form $[-a, a]$. (Hint: to do this you will need to estimate the size of $x^n/n!$ when $|x| \leq a$ and $n \to \infty$. For example, can you prove that $n! > (2a)^n$? How large must n be in terms of a?)

Deduce that $R(E)$, the radius of convergence of $E(x)$, is infinite. Deduce from Theorem 6.5.7 in the textbook that the derivative of $E(x)$ may be computed term by term. What is it?

Problem 10.2. Prove that $E(x) \cdot E(-x) = 1$. (Hint: You are trying to prove that the function given by the left-hand side is *constant*.) Deduce the formula $E(-x) = 1/E(x)$.

Problem 10.3. Prove that $E(x)$ is positive for all $x \geq 0$. Deduce, from Problem 10.2 that $E(x)$ is positive for all real x. Prove that $E(x)$ is strictly increasing for all real x.

Problem 10.4. Now fix a real number $a \in \mathbb{R}$. Let $f(x) = E(x + a)$ and let $g(x) = E(x) \cdot E(a)$. Last week I attempted to sketch a proof that $f(x) = g(x)$ for all real x, using the Lagrange Remainder Theorem.

Prove that $f(x) = g(x)$ for all real x. (Hint: let $h(x) = f(x)/g(x)$. You are trying to prove that h is a *constant* function. Note that h is well-defined because $E(x)$ is never zero, by Problem 10.3.)

Problem 10.5. Prove that $2.7 < E(1) < 2.8$. (As all terms of $E(1)$ are positive the lower bound is easy. The upper bound will require more thought.)

Problem 10.6. Define

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots, \quad C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Prove that both $S(x)$ and $C(x)$ have infinite radius of convergence. Prove that $S'(x) = C(x)$ while $C'(x) = -S(x)$ using Theorem 6.5.7.

Problem 10.7. We would like to obtain the identity:

$$[S(x)]^2 + [C(x)]^2 = 1.$$

One way to do this would be to actually multiply out the squares, using induction, and add. However, we would need to understand when the square of a power series converges. This requires Abel summation or some similar idea. (See the end of Chapter 8 of Hardy's book *A course of pure mathematics*.)

Prove the identity holds. (Hint: You are being asked to prove that $[S(x)]^2 + [C(x)]^2$ is *constant*.)