Math 311: Section 3.

Solutions for final homework.

These are solutions (or at least hints) for the homework problems on page 199, numbers 7.4.4 and 7.4.6, and page 202, number 7.5.4.

Solution 14.1. Prove or give a counterexample.

- **If $\int |f|$ is integrable on $[a, b]$ then f is also integrable on this set.**
  This is false. Cosider the function $f(x) = d(x) - \frac{1}{2}$ where $d(x)$ is Dirichlet's function. See page 7 of Abbott's book.
- **Assume g is integrable and $g \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$ then $\int g > 0$.**
  This is also false. Think about Exercise 7.3.5.
- **If g is continuous on $[a, b]$ and $g \geq 0$ with $g(x_0) > 0$ for at least one point $x_0 \in [a, b]$ then $\int_a^b g > 0$.**
  This is true. Suppose that $g(x_0) = c > 0$. Let $\epsilon = c/2$. As g is continuous there is a $\delta > 0$ so that $|x - x_0| < \delta$ implies $|g(x) - g(x_0)| < c/2$. It follows (check!) that $|x - x_0| < \delta$ implies $g(x) > c/2 > 0$.
  Now you should prove that $\int_a^b g \geq \frac{c\delta}{2}$. For example, consider any partition that contains the points $x_0 \pm \frac{\delta}{2}$. Give a lower bound for the lower sum.
- **If $\int_a^b f > 0$ there is an interval $[c, d] \subset [a, b]$ and a $\delta > 0$ such that $f(x) > \delta$ for all $x \in [c.d]$.**
  This is also true. Choose a partition P so that the lower sum $L(f, P)$ is positive. (Why does such a partition exist?) Recall that $L(f, P) = \sum m_k(x_k - x_{k-1})$, where $m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$. Since $L(f, P)$ is positive at least one summand, say $m_j(x_j - x_{j-1})$, must be positive. So m_j is positive. Thus for all $x \in [x_{j-1}, x_j]$ we have $f(x) \geq m_j > 0$. Set $\delta = \frac{m_j}{2}$ and we are done.

Solution 14.2. These questions focus on the discussion preceding Theorem 7.4.4.

- **Produce an example of a sequence $f_n \to 0$ pointwise on $[0, 1]$ where $\lim_{n\to\infty} \int_0^1 f_n$ does not exist.**
  Consider the functions $f_n(x) = \begin{cases} 0, & 0 = x \\ n^2, & 0 < x < 1/n \\ 0, & 1/n \leq x. \end{cases}$
- **Produce another example (if necessary) where $f_n \to 0$ and the sequence $\int_0^1 f_n$ is unbounded.**
  It is not necessary.
- **Is it possible to construct each f_n to be continuous in the questions above?**
  Yes – graph f_3 as described above and think about how you would alter it to make it continuous.
- **Does it seem possible to construct the sequence (f_n) to be uniformly bounded?**

No. Suppose that $|f(x)| < M$ for all $x \in [0,1]$ and f is integrable. Then by Theorem 7.4.2 $|\int_0^1 f| \leq M$. That is, a collection of uniformly bounded functions have uniformly bounded integrals.

Solution 14.3. Let $H(x) = \int_0^x \frac{dt}{t}$ where we consider only $x > 0$.

- $H(1) = 0$. $H'(x) = 1/x$ by Theorem 7.5.1.
- Note that $H'(x) = 1/x > 0$ for all $x > 0$. Then H is increasing by the Mean Value Theorem (Theorem 5.3.2). See Exercise 5.3.7.
- Let $g(x) = H(cx)$ and let $h(x) = H(c)+H(x)$. We notice that $g(1) = H(c)$ and also $h(1) = H(c) + H(1) = H(c)$, as $H(1) = 0$. Thus $g(1) = h(1)$. We now differentiate both functions using the chain rule: $g'(x) = c/cx = 1/x$ while $h'(x) = 0+1/x = 1/x$. Thus g and h differ by a constant, again by the Mean Value Theorem. See Corollary 5.3.4. We deduce that $g(x) = h(x)$ for all $x > 0$ because they agree at a point.