Math 311: Section 3.

Hints for some Chapter 6 problems.

Solution 15.1 (6.2.1)**.** Let $f_n(x) = \frac{nx}{1+nx^2}$.

- To find the pointwise limit means "fix x and let n go to infinity." We find, if $x > 0$, that

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim \frac{x}{\frac{1}{n}+x^2} = x/x^2 = 1/x$$

  If $x = 0$ then the limit does not exist. For future use, let $f(x) = 1/x$.
- Let $x_n = 1/n$ for $n \in \mathbb{N}$. Then $f(n_n) = n$ while $f_n(x_n) = \frac{1}{1+1/n}$. Now prove that the existence of such a sequence contradicts uniform convergence.
- As the $x_n \to 0$ the "proof" given above shows that the functions f_n do not converge uniformly on *any* open interval with zero as an endpoint.
- Yes. Compute the difference $|f_n(x) - f(x)|$ and prove that it is bounded above by $1/n$, independently of the value of $x \in (1, \infty)$.

Solution 15.2 (6.2.11)**.** Assume that (f_n) and (g_n) are uniformly convergent sequences of functions. Let f and g be the limit functions.

- Bound the difference $|(f_n + g_n) - (f + g)|$. The triangle inequality will be useful.
- Consider the sequences $f_n(x) = g_n(x) = x + \frac{1}{n}$. Show that these converge uniformly on $\mathbb{R}$ to the function $f(x) = x$. Now consider $(f_n\, g_n)(x) = x^2 + \frac{2x}{n} + \frac{1}{n^2}$. How close is this to the function $1/x^2$? Is the convergence uniform?
- If all functions involved are uniformly bounded (*i.e.*, there is a single $M \in \mathbb{R}$ bounding all the functions) then the limit functions f and g are also bounded. To prove that the convergence is uniform again bound the difference $|f_n g_n - fg|$. Add and subtract fg_n inside the absolute value and think about how you could use the triangle inequality.

Solution 15.3 (6.3.2)**.** Consider the sequence of functions $g_n(x) = x^n/n$ defined on $[0, 1]$.

- The pointwise limit of (g_n) is the function $g(x) = 0$. As $|g_n(x)| \le 1/n$ in the domain of interest, the convergence is uniform.

  Here is a complete proof, directly following the definition of uniform convergence: Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $N > 1/\epsilon$. (Archimedean Property) It follows that $\epsilon > 1/N$. Note that, for any $n \ge N$ we have $1/N \ge 1/n$ and thus $\epsilon > 1/n$. Now compute $|g_n(x) - g(x)| = |x^n/n - 0| = x^n/n \le 1/n$, this last because $x \in [0, 1]$. We conclude that $|g_n(x) - g(x)| < \epsilon$, as desired.

  Finally, since $g(x) = 0$ is a constant function $g'(x)$ exists for $x \in [0, 1]$ and equals zero.

- Note that $g_n'(x) = x^{n-1}$. So (g_n') converges pointwise to the function $h(x)$ which is zero for all $x \in [0, 1)$ while $h(1) = 1$. This function h is not continuous. As the functions g_n' are continuous we deduce from Theorem 6.2.6 that the convergence $g_n' \to h$ is *not* uniform.

  Note also that g' disagrees with h at a single point. This is not a contradiction to Theorem 6.3.1.

Solution 15.4 (6.4.1)**.** Apply the Cauchy criterion (Theorem 6.4.4) with $n = m + 1$.

Solution 15.5 (6.4.5)**.** Let

$$f(x) = \sum \frac{\sin(kx)}{k^3}$$

This usually referred to as a *Fourier series.*

- This is a direct application of Theorem 6.4.3, several uses of the M-test (6.4.5), and the so-called "p-test": $\sum \frac{1}{k^p}$ converges if and only if $p > 1$ which we proved in workshop. The continuity of the derivative follows from Theorem 6.2.6.
- The only honest answer is "No." By the above we have $f'(x) = \sum \frac{\cos(kx)}{k^2}$. The term-by-term derivative of this, and thus our only candidate for a second derivative of f, is $h(x) = -\sum \frac{\sin(kx)}{k}$. We cannot apply the M-test here, as the harmonic series diverges. Without the M-test we cannot show that h converges uniformly and so cannot use Theorem 6.4.3.

  In fact, $f'(x)$ is a Fourier expansion of the 2π-periodic function which takes the value $\frac{1}{4}\left((x - \pi)^2 - \frac{\pi^2}{3}\right)$ for $x \in [0, 2\pi]$. It follows that f' is differentiable everywhere except at $2\pi \cdot \mathbb{Z} \subset \mathbb{R}$. Thus the original function f is *not* twice-differentiable.

This last problem raises an interesting question: does $h(x) = -\sum \frac{\sin(kx)}{k}$ converge uniformly in the interval $(0, 2\pi)$? Nothing we said above rules this out. However, this is again not case – the problem is the *Gibbs phenomenon.* This is distantly related to the frankly jaw-dropping identity

$$\int_0^\infty \frac{\sin(t)}{t} dt = \frac{\pi}{2}$$

which seems like as good a place as any to end.