Homework Solutions

Debajyoti Nandi

April 1, 2005

- **Problem 1 (p119, #4.4.1)**
  (a) Show that $f(x) = x^3$ is continuous on all of $\mathbb{R}$.
  (b) Argue, using theorem 4.4.6, that f is not uniformly continuous on $\mathbb{R}$.
  (c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

  (a) By Theorem 4.3.4, we know that products of continuous functions are continuous. Hence, it is enough to show that the function $g(x) = x$ is continuous on all of $\mathbb{R}$. Given any $\epsilon < 0$, choose $\delta = \epsilon$. Then for any $x, y \in \mathbb{R}$, $|x - y| < \delta \Rightarrow |g(x) - g(y)| = |x - y| < \epsilon$.

  (b) Choose $x_n = n$, and $y_n = n - 1/n$. Then $|x_n - y_n| \to 0$, but

  $$|f(x_n) - f(y_n)| = 3n - 1/n(3 - 1/n^2) \to \infty.$$

  Thus, by Theorem 4.4.6, the function f is not uniformly continuous on $\mathbb{R}$.

  (c) Let $A \subset \mathbb{R}$ be any bounded subset. Then $\overline{A}$, the closure of A in $\mathbb{R}$, is compact. By Theorem 4.4.8, f is uniformly continuous on $\overline{A}$, and hence on any subset of $\overline{A}$. In particular, f is uniformly continuous on A.

- **Problem 2 (p119 #4.4.4)** *Show that if f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$.*

  Let $m = \inf_{a \leq x \leq b} f(x)$. Since $[a, b]$ is compact, and f is continuous, f attains its minimum m, say at $x_0 \in [a, b]$. Therefore, $m = f(x_0) > 0$. Hence $|1/f(x)| = 1/f(x) \leq 1/m$ for all $a \leq x \leq b$.

- **Problem 3 (p120 #4.4.9)**
  (a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is unifomly continuous on A.
  (b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

  (a) Assume that f is Lipschitz with Lipschitz constant M. Now, given $\epsilon > 0$, choose $\delta = \epsilon/M$. Then we have,

  $$|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq M|x - y| < \epsilon,$$

  as desired.

  (b) The converse is not true.

  Take $f : [0, 1] \to \mathbb{R}$, given by $f(x) = \sqrt{x}$. Then f is uniformly continuous on $[0, 1]$, since it is continuous on a compact set. But f is not Lipschitz. Given any $M > 0$, choose x such that $0 < x < 1/M^2$, and $y = 0$. Then $\frac{|f(x)-f(y)|}{|x-y|} = 1/\sqrt{x} > M$.

- **Problem 4 (p124 #4.5.3)** *Is there a continuous function on $\mathbb{R}$ with range $f(\mathbb{R}) = \mathbb{Q}$?*

  We know that $\mathbb{R}$ is connected (by completeness), whereas $\mathbb{Q}$ is not. To see that $\mathbb{Q}$ is not connected notice that $\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \pi)) \cup (\mathbb{Q} \cap (\pi, \infty))$. Hence, by Theorem 4.5.2, there is no continuous function f with $f(\mathbb{R}) = \mathbb{Q}$.

([Saul] Here is a morally equivalent proof which avoids connectedness. Suppose f is a function with $f(\mathbb{R}) = \mathbb{Q}$. Then f takes on the values 3 and 4 but not the value π. Deduce from the IVT that f is not continuous.)

- **Problem 5 (p124 #4.5.7)** *Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.*

  We have $0 \leq f(x) \leq 1$ for all $0 \leq x \leq 1$. Consider a function $g : [0, 1] \to \mathbb{R}$, given by $g(x) = f(x) - x$. Then g is continuous on $[0, 1]$. Now $g(0) = f(0) \geq 0$, but $g(1) = f(1) - 1 \leq 0$. Hence, by the Intermediate Value Theorem (Theorem 4.5.1), there exists $x_0 \in [0, 1]$ such that $g(x_0) = 0$. Therefore, we have $f(x_0) = x_0$.