1 Introduction

Graphs are among the simplest objects in mathematics! Usually they are used to record a single kind of yes/no relationship between a collection of objects. Here is an example:

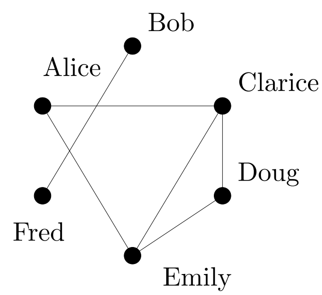

Figure 1: Who invited Bob and Fred?

In this example, we have a bird's eye view of a party. The black dots represent the people at the party, and the lines represent *pairs* of people who know each other. Clearly, Bob and Fred know each other, but neither of them knows anybody else at the party.

Abstracting the example just a bit we make the following definition:

Definition 1.1. A *simple graph* G is a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*. An edge is an unordered pair of vertices.

In the example shown in Figure 1 there are six vertices. Namely,

$$V(G) = \{\text{Alice}, \text{Bob}, \text{Clarice}, \text{Doug}, \text{Emily}, \text{Fred}\}.$$

There are also six edges. Using the abbreviation A for Alice, B for Bob, and so on we have

$$E(G) = \{\{A, C\}, \{A, E\}, \{B, F\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

Notice that certain questions are easier to answer if you look at the picture and harder to answer just from the information $V(G)$ and $E(G)$.

For example, "does Alice know Doug?" In the picture we just need to make sure that no edge connects the A vertex to the D vertex. Without the picture we would have to check all of $E(G)$ to make sure that the pair $\{A, D\}$ does not appear. Here is another kind of information: how many triples of mutual friends are there at the party? (A triple of mutual friends is a set of three people, all of whom know each other.) Staring at the graph we realize that a triple of mutual friends is just a triangle! There are two triples: $\{A, C, E\}$ and $\{C, D, E\}$. I invite you to check this just using the information encoded in the list $E(G)$.

Here is yet another kind of information: a *triple of mutual strangers* is a set of three people, all of whom don't know each other.

Exercise 1.2. How many triples of mutual strangers are there at the party?

2 Complete graphs and complements

At a very unfriendly party, nobody knows anyone else. At a very friendly party indeed, we see that everybody knows everybody else.

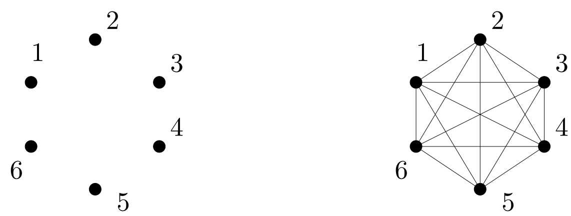

Figure 2: The null graph N_6 and the complete graph K_6.

These two extremes are called the *null* graph on the vertex set V and the *complete* graph on V, respectively. To be precise:

Definition 2.1. Fix a vertex set V of size n. The null graph on V has $E = \emptyset$ and is denoted by N_n. The complete graph on V has E as large as possible and is denoted by K_n. Said another way: $E(K_n)$ contains all possible edges.

Null graphs are not terribly interesting. Complete graphs are a bit more interesting.

Exercise 2.2. How many edges does K_6 have? How many triangles? How many edges does K_n have? How many triangles? (Do this for $n = 1, 2, 3, 4, 5, \ldots$ and look for a pattern.)

Suppose now that we are given a graph, say the one shown in Figure 1. Let's call this graph P, for "party". Note that P has six vertices and six edges. We can form a new graph, $\bar{P}$ which tells us something different about the party – namely take $\bar{P}$ to have the same vertex set as P but connect two people if they *don't* know each other. I'll draw this "strangers graph" with dotted lines. Next to it I'll draw the friends graph and the strangers graph together:

Figure 3: Dotted lines indicate strangers. Solid lines indicate friends.

It is easier to answer Exercise 1.2 now we have drawn the strangers graph. Also, notice that the friends graph P with the strangers graph $\bar{P}$ taken together is a complete graph. This is not an accident!

Definition 2.3. Fix a simple graph G. The *complement* of G, called $\bar{G}$, is the simple graph with the same vertices as G but where two vertices are connected by an edge if and only if they are *not* connected in G.

Suppose that G and G' are two graphs with the same underlying vertex set (that is, $V(G) = V(G')$). We can form $G \cup G'$, the *union* of G and G', by taking $V(G \cup G') = V(G)$ and $E(G \cup G') = E(G) \cup E(G')$. For example:

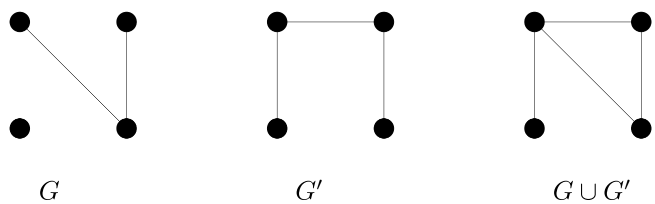

Figure 4: The third graph is the union of the first two.

Lemma 2.4. *The union of G and $\bar{G}$ is a complete graph.* □

Exercise 2.5. Prove Lemma 2.4.

3 Ramsey theory

Staring at the right-hand side Figure 3 for a few moments we see that there is both a solid triangle (a triple of friends) and also a dotted triangle (a triple of strangers). Must this always happen? Is there always both a triangle of friends and a triangle of strangers? Well, looking at the very unfriendly and very friendly parties shown in Figure 2 we see that this is not the case. The null graph N_6 has a triple of strangers and no triple of friends. For the complete graph K_6 the situation is reversed.

We therefore make a new conjecture: every party has *either* a triple of friends *or* a triple of strangers. Is this new conjecture true? Well, a bit of thought shows that this is not always the case. See Figure 5.

Figure 5: A party with no triangles of either type.

This leads us to a new conjecture (which is actually true and so is called a theorem):

Theorem 3.1. *For any simple graph G with at least six vertices either G contains a triangle or $\bar{G}$ contains a triangle. (That is, "any party of at least six people has either a triple of friends or a triple of strangers.")*

Before discussing the proof, I invite you to draw a few random six vertex graphs, and check that the theorem holds for those graphs.

Exercise 3.2. Do it!

Before we begin the proof, we introduce one more piece of notation:

Definition 3.3. Suppose that $v \in V(G)$ is a vertex of a graph G. The *degree* of v, denoted $\deg(v)$, is the number of edges touching v.

As an example, check that the degrees of all of the vertices of the graph in Figure 1 are $(\deg(A), \ldots, \deg(F)) = (2, 1, 3, 2, 3, 1)$. Here, now, is a sketch of the proof of Theorem 3.1:

Proof. Fix attention on a graph G with exactly six vertices. Pick one of the vertices and label it with the letter a. Suppose for the moment that $\deg(a) \geq 3$. Attach the labels b, c, and d to three of the vertices which a is connected to.

Now, if there is any edge connecting any of the pairs $\{b, c\}$, $\{c, d\}$, or $\{d, b\}$ then there is a triangle in G and the theorem holds. If none of the edges $\{b, c\}$, $\{c, d\}$, or $\{d, b\}$ are present in G, then *all* of them are present in $\bar{G}$. Again, the theorem holds.

What happens if our assumption that $\deg(a) \geq 3$ is false? What happens if G has more than 6 vertices? I leave it to you. □

Question 3.4. How many people must you invite to your party to guarantee that there are either four mutual friends *or* four mutual strangers? (This is very hard. An easier question would be: How many people must you invite to your party to guarantee that there are either three mutual friends *or* four mutual strangers?)