1 Graphs and isomorphism

Last time we discussed *simple graphs*:

Definition 1.1. A *simple graph* G is a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*. An edge is an unordered pair of distinct vertices.

This is essentially the correct definition. However there are two things forbidden to simple graphs – no edge can have both endpoints on the same vertex and no pair of vertices can have multiple edges between them. The general definition of a graph allows both of these behaviors:

Definition 1.2. A *graph* G is a set $V(G)$ of *vertices* and a family $E(G)$ of *edges*. An edge is an unordered pair of vertices.

Note the differences. Instead of a set of edges we now have a *family* of edges, so *multiple* edges are allowed. Instead of an edge being an unordered pair of distinct vertices we allow repeated vertices, so an edge can meet one vertex twice. These special edges are called *loops*. See Figure 1.

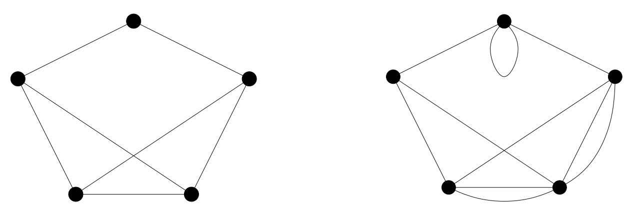

Figure 1: The graph of the left is simple. The graph on the right is not.

We often only consider only simple graphs. However, non-simple graphs do occur in real-life – consider a road-map where there are many roads connecting two cities.

We now turn to the very important concept of *isomorphism* of graphs:

Definition 1.3. Two graphs G and H are *isomorphic* if there is a bijection $f\colon V(G) \to V(H)$ so that, for any $v, w \in V(G)$, the number of edges connecting v to w is the same as the number of edges connecting $f(v)$ to $f(w)$.

Note that we do not assume that $v \neq w$ in the definition. Thus it follows that the number of loops at v must also equal the number of loops at $f(v)$.

The motivating idea here is that if G and H are isomorphic then they are identical, "in every way that matters." For example, the way that G and H are drawn does not matter. The labels we put on the vertices of G and H does not matter. What *does* matter, loosely, is the "shape" of G and H.

To prove that two graphs G and H *are* isomorphic is simple: you must give the bijection f and check the condition on numbers of edges (and loops) for all pairs of vertices $v, w \in V(G)$. To prove that G and H are *not* isomorphic can be much, much more difficult. What is required is some property of G where

- H does not have the property and
- *every* graph isomorphic to G *does* have the property.

For example we have:

Lemma 1.4. *If G is not simple and H is simple then G is not isomorphic to H.*

Proof. Let's prove this by contradiction: suppose that $f\colon V(G) \to V(H)$ is the bijection giving the isomorphism. There are two cases. Suppose first that G has a multiple edge, say between v and w. Then H must have a multiple edge between $f(v)$ and $f(w)$. But H is simple and so has no multiple edges. This is a contradiction.

To deal with the other case: suppose that G has a loop at v. Then H must have a loop at $f(v)$. But H is simple and has no loops. Again, this is a contradiction. □

This lemma and its proof is a model for several of the arguments which we only sketch below. To repeat: to prove that G and H are *not* isomorphic it suffices to find a property of graphs which is preserved by isomorphism but which G and H do not share. There are many such properties: this indicates the subtle nature of isomorphism.

2 Degrees

Consider the following two graphs shown in Figure 2.

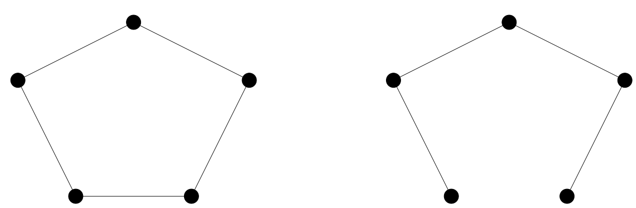

Figure 2: The graph on the left, C_5, is the *cycle* on 5 vertices. The graph on the right, P_5, is the *path* on 5 vertices. Are they isomorphic?

In general, we have C_n, the cycle on n vertices and P_n, the path on n vertices. The question is: Is C_n isomorphic to P_n?

The answer is clearly no: the reason is that the path graph has two vertices with only one edge each. But in the cycle graph graph every vertex has exactly two edges. So no bijection exists with the desired properties.

To make this precise we recall that the *degree* of a vertex $v \in V(G)$ in the number of ends of edges arriving at v. (So loops count double!) We use the notation $\deg(v)$ to denote the degree or *valency* of v.

Definition 2.1. Given a graph G the *degree sequence* of G is the list of all degrees of vertices in G, in non-increasing order.

For example, the degree sequence of C_5 is $(2, 2, 2, 2, 2)$ and that of P_5 is $(2, 2, 2, 1, 1)$. Notice that the length of the degree sequence of G is the same as the number of vertices of G.

Exercise 2.2. Prove that if G and H are isomorphic then G and H have the same degree sequence. (Consequently, if G and H have different numbers of vertices then they are not isomorphic.)

Here is the *handshaking lemma*:

Lemma 2.3. *For any graph G we have:*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Here $|E(G)|$ is the number of edges of G. □

This follows because every edge has two endpoints. As a pair of corollaries we have first: *the sum of any degree sequence is always even* and second: *every graph has an even number of vertices with odd degree.* You might want to check these surprising claims for a few examples!

We end this section with a few more pieces of terminology. A vertex of G is *isolated* if it has degree zero. A vertex of G is an *end-vertex* (also called a *leaf*) if it has degree one.

An important class of examples are the *regular* graphs. These are graphs where the degree sequence is constant. For example, if $\deg(v) = 3$ for all $v \in V(G)$ then we call G a *three-regular* or *cubic* graph. See Figure 3 for an example of a cubic graph.

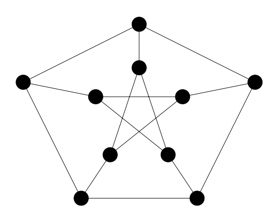

Figure 3: The Peterson graph.

Exercise 2.4. Suppose that G is cubic and has n vertices. Is n even or odd? How many edges does G have?

Exercise 2.5. Suppose that G is a simple graph and $\bar{G}$ is the *complement* of G. What is the relationship between the degree sequences of G and $\bar{G}$? (This is not a simple yes/no question. You will have to explore several examples in order to understand what is going on. For example, what is the degree sequence of a complete graph? Of a null graph? What are the degree sequences of the graphs in Figure 2.9 on page 11 of the book?)

3 Connectedness

Another important invariant of a graph is *connectedness*. Before giving a long-winded definition here is a simple example. Let $G = C_6$ and let H be the disjoint union of two copies of C_3. See Figure 4.

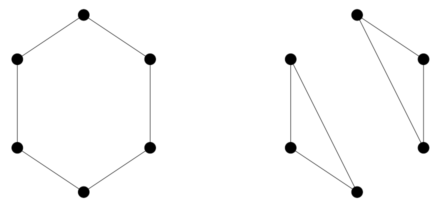

Figure 4: The graphs G and H have the same degree sequences.

It is easy to see that both G and H are 2-regular and both have six vertices. Thus G and H have identical degree sequences. So why are G and H not isomorphic? Before reading on: can you *prove* that G and H are not isomorphic?

Here is the relevant definition:

Definition 3.1. A graph G is *disconnected* if there is a partition of $V(G)$ into two sets A and B so that no edge in $E(G)$ connects a vertex of A to a vertex of B. A graph is *connected* if no such partition exists.

In general we will concentrate on connected graphs. Certainly, using a disconnected graph as a transportation network would be a great mistake! It would give rise to a kind of "you can't get there from here" situation.

We finish with a few exercises.

Exercise 3.2. Find graphs G and H with the following properties:

- Both have exactly five vertices.
- Both are simple and connected.
- G is not isomorphic to H.

Figure 2.9 on page 11 of the book may be useful.

Exercise 3.3. Give a list of graphs where

- Every graph on the list is simple, connected, and has exactly four vertices.
- The list is *repetition-free*: no pair of graphs on the list are isomorphic.
- The list is *exhaustive*: every simple connected graph with four vertices appears somewhere on the list.

(If this is too hard start by making such a list for graphs with three vertices. If this is too easy – can you do the same for graphs with five vertices? Note that it is not enough to point to Figure 2.9 on page 11 of the book and say "Done." You must actually prove that your lists are repetition-free and exhaustive.