1 Connected simple graphs on four vertices

Here we briefly answer Exercise 3.3 of the previous notes. Please come to office hours if you have any questions about this proof.

Theorem 1.1. *There are exactly six simple connected graphs with only four vertices. They are listed in Figure 1.*

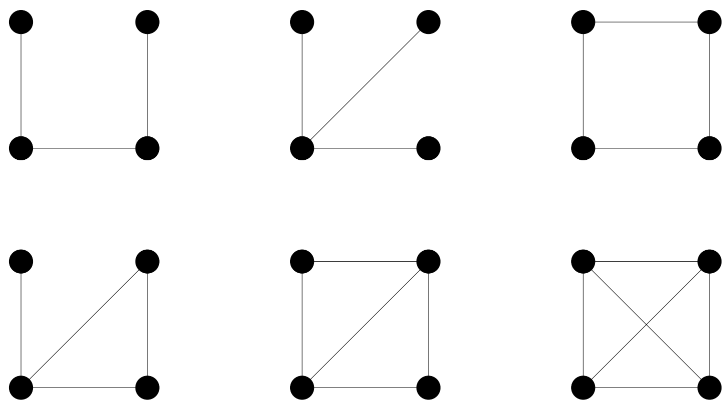

Figure 1: An *exhaustive* and *irredundant* list. We order the graphs by number of edges and then lexicographically by degree sequence.

The proof is arranged around first, the number of edges and second, the idea of the degree sequence. We begin with a few observations. Suppose that G is simple, connected, and $|V(G)| = 4$. It follows that the degree sequence of G has length four. Also, all degrees are at most three (as G is simple) and at least 1 (as G is connected so it has no isolated vertices).

If G is to be connected, then $|E(G)| \geq 3$. As G is simple, $|E(G)| \leq 6$. We deal with the possible values of $|E(G)|$, namely 3, 4, 5, and 6 in turn.

The *handshaking lemma* implies that the sum of the degrees is even. From all of this it follows that the lexicographically first possible degreee sequence is $(1, 1, 1, 1)$. However any graph with this degree sequence has only two edges.

$V(G) = \{x, y, z, w\}$. Then x is incident to only one edge. Say the other endpoint of this edge is y (as the other possibilities are similar). Then neither x nor y has any other edge incident and we conclude that G is not connected.

The next possible degree sequence for G is $(2, 2, 1, 1)$. We take $V(G) = \{x, y, z, w\}$. Suppose that the degrees of x, y, z, and w are non-increasing. Thus both z and w are degree one vertices, ie *leaves*. If z and w are connected to each other then G is not connected, a contradiction. If both z and w are connected to x, say, then y must have a loop, a contradiction. We conclude that x and y share an edge and G is isomorphic to the first graph of Figure 1.

Next we have $(3, 1, 1, 1)$. If G is to be simple, then x is connected to each of the other three vertices exactly once. We obtain the third graph of Figure 1.

We now increase the number of allowed edges by one – the sequences must sum to eight instead of six. The next degree sequence is $(2, 2, 2, 2)$. Prove to your own satisfaction that any graph with degree sequence $(2, 2, 2, \ldots, 2)$ must be a union of cycles. It will follow that G is isomorphic to C_4.

Next is $(3, 2, 2, 1)$. Again, x is connected to each of the other three vertices exactly once. This accounts for three edges. We must add exactly one more. Check that regardless of which two vertices we connect (y to z, z to w, or w to y) we obtain the forth graph.

We now deal with $|E(G)| = 5$ or 6. The next sequence is $(3, 3, 2, 2)$. Clearly x is connected to y, z, and w. Likewise y is connected to x, z, and w. This gives the correct degree sequence and we have the fifth graph. The sequence $(3, 3, 3, 1)$ does not correspond to a simple graph. Finally we have the sequence $(3, 3, 3, 3)$ which gives the complete graph K_4. We are done.

2 Paths

After all of that it is quite tempting to rely on degree sequences as an infallable measure of isomorphism. However, *that* would be a mistake, as we shall now see.

In our first example, Figure 2, we have two connected simple graphs, each with five vertices. You should check that the graphs have identical degree sequences.

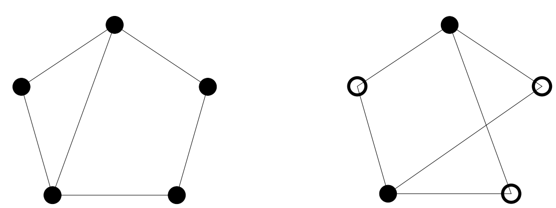

Figure 2: A pair of five vertex graphs, both connected and simple. Both have the same degree sequence. Are they isomorphic?

However, the graphs are not isomorphic. To prove this, notice that the graph on the left has a triangle, while the graph on the right has no triangles. Similarly, in Figure 3 below, we have two connected simple graphs, each with six vertices, each being 3-regular. It follows that they have identical degree sequences. Again, the graph on the left has a triangle; the graph on the right does not. (Check!).

It is easy to see what is going on: the two graphs are different because one has some kind of "path" that the other graph does not have. We formalize this notion as follows.

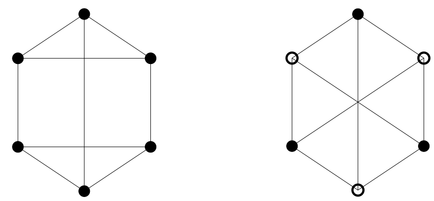

Figure 3: The graphs $C_3 \times I$ and $K_{3,3}$. Both are connected, simple, and three-regular. Are they isomorphic?

Definition 2.1. Suppose that G is a graph. A sequence of edges

$$e_1, e_2, \ldots, e_i, e_{i+1}, \ldots, e_n$$

is a *walk* in G of length n if there are vertices $v_0, v_1, \ldots, v_n$ so that for all i the edge e_i connects v_{i-1} to v_i.

This is somewhat formal – if G is simple it is equivalent to think of a walk as just a sequence of adjacent vertices. It all cases it is ok to think of a walk as a way of tracing your pencil over the edges of the graph, starting at v_0, ending at v_n, without lifting the pencil.

If we never repeat an edge in the walk then we call it a *trail*. If we never repeat a vertex in the walk (except for possibly allowing $v_0 = v_n$) then we call it a *path*. If in fact $v_0 = v_n$ then we call the walk, trail, or path *closed*. Another name for a closed path is a *cycle*.

So what we were calling a triangle above now has another name – it is a cycle of length three. Counting cycles in G gives isomorphism invariants. For example, if G has 3 triangles and H has 4 then G is not isomorphic to H.

Here is a simple lemma which we will need below:

Lemma 2.2. *If G contains a closed walk of odd length, then G contains a cycle of odd length.*

Proof. We induct on the length, n, of the given closed walk $W = (v_0, v_1, \ldots, v_n)$. If $n = 1$ then G contains a loop and we are done. Suppose now that $n \geq 3$. If W is a path we are done. So suppose that $v_i = v_j$ where $i < j < n$.

If $j - i$ is odd then the walk

$$W' = (v_i, v_{i+1}, \ldots, v_j)$$

is a closed walk of odd length, shorter than n. If $j - i$ is even then the walk

$$W'' = (v_0, v_1, \ldots, v_{i-1}, v_i, v_{j+1}, v_{j+2}, \ldots, v_n)$$

is a closed walk of odd length, shorter than n. By induction, the proof is complete. $\square$

Definition 2.3. We say that a graph is *Eulerian* if there is a closed trail which vists every edge of the graph exactly once.

Definition 2.4. We say that a graph is *Hamiltonian* if there is a closed path walk which vists every vertex of the graph exactly once.

We will discuss these in greater detail next week.

3 Making small examples

There are many ways of creating new graphs from old. Here we concentrate on making G smaller.

Definition 3.1. If G is a graph then any graph appearing in G is called a *subgraph* of G. For example, if e is an edge of G then we can delete e from G to form the graph $G - e$. If v is a vertex of G then we can delete v and all edges incident to v to form $G - v$. Every subgraph of G is obtained by deleting some sequence of edges and vertices of G. If H is a subgraph of G we write $H \subset G$.

Exercise 3.2. Suppose that G is a simple graph on n vertices. Show that G appears as a subgraph of K_n, ie $G \subset K_n$.

4 Return to connectedness

Recall that a graph G is *disconnected* if there is a partition $V(G) = A \cup B$ so that no edge of $E(G)$ connects a vertex of A to a vertex of B. If there is no such partition, we call G *connected*. We now use paths to give a characterization of connected graphs.

Theorem 4.1. *A graph G is connected if and only if for every pair of vertices v and w there is a path in G from v to w.*

Proof. We begin with the forward direction. Fix a vertex $v \in V(G)$. We define:

$$A_v = \{w \in V(G) \mid v \text{ and } w \text{ are connected by a path in } G\}.$$

Let $B_v = V(G) - A_v$. We will show that B_v is empty and thus the conclusion holds. Suppose, for a contradiction, that B_v is not empty. As G is connected there is an edge e connecting some $w \in A_v$ to some $u \in B_v$. There is path P from v to w which does not mention u (because $u \notin A_v$). So append the edge e to form a longer path $P \cup e$, connecting v to u. It follows that $u \in A_v$! This is a contradiction.

To prove the reverse direction: Fix a partition $V(G) = A \cup B$. We must find an edge e running from a vertex of A to a vertex of B. To this end, choose any $v \in A$ and $w \in B$. By assumption there is a path from v to w. Label the vertices of the path

$v_0, v_1, v_2, \ldots, v_n$ so that $v = v_0$, $w = v_n$, and v_i and v_{i+1} are consecutive vertices of the path. Then there is a vertex with smallest possible label, say v_k, with $v_k \in B$. Now, $v_k \neq v$, as $v \in A$. So $k > 0$. Thus v_{k-1} is in A, $v_k \in B$, and v_{k-1} and v_k are connected by an edge. □

It is easy to adapt the proof just give to show that: *G is connected if and only if there is a walk connecting every pair of vertices of G.*

Disconnected graphs break naturally into smaller pieces: Suppose H is a connected subgraph of G. Suppose also that all subgraphs $H' \subset G$ containing H are disconnected. Then we call H a *component* of G. Put another way: if you draw a disconnected graph and then draw circles around each of the pieces then you have circled the components.

There another two common definitions relating to connectedness.

Definition 4.2. If G is connected and $G - e$ is disconnected then we call the edge e a *bridge*.

Definition 4.3. If G is connected and $G - v$ is disconnected then we call the vertex v a *cut-vertex*.

For example, every edge of the path graph P_n is a bridge but no edge of the cycle C_n is. All of the vertices of P_n having degree two are cut vertices. A leaf is *never* a cut vertex. A very important class of graphs are the *trees*: a simple connected graph G is a *tree* if every edge is a bridge. (Equivalently, if every non-leaf vertex is a cut vertex.)

The idea of a bridge or cut vertex can be generalized to sets of edges and sets of vertices. We will develop such extensions later in the course.

5 Making large examples

Now we may focus on making big examples.

Definition 5.1. Suppose that G is a graph. Choose a point v which is *not* a vertex of G. Form the graph G' by adding v to the vertex set of G and by adding edges between v and every vertex of G. We call G' the *cone* on G.

Note that the graph G naturally appears as a subgraph of the cone G'. As an example, the cone on a cycle is called a *wheel*.

Definition 5.2. Suppose that G and H are graphs. We form the *product* of G and H, called $G \times H$, taking $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = E(G) \times E(H)$. The product of two edges is a *square*: $\{v, w\} \times \{a, b\}$ gives the four edges $\{(v, a), (w, a)\}$, $\{(w, a), (w, b)\}$, $\{(w, b), (v, b)\}$, and $\{(v, b), (v, a)\}$ in the product graph.

The simplest example of this is taking the product of a graph G with the *interval*: the simple graph I with exactly two vertices and exactly one edge. For example $Q^2 = I \times I$ is the *square*; which is identical to the 4-cycle C_4. $Q^3 = I \times C_4$ is the *cube*, and $Q^4 = I \times \text{cube}$ is the *hypercube* or *tesseract*. The hypercube is often called the "4-cube". The ordinary cube Q^3 would then be called the "3-cube", and so on. See Figure 4.

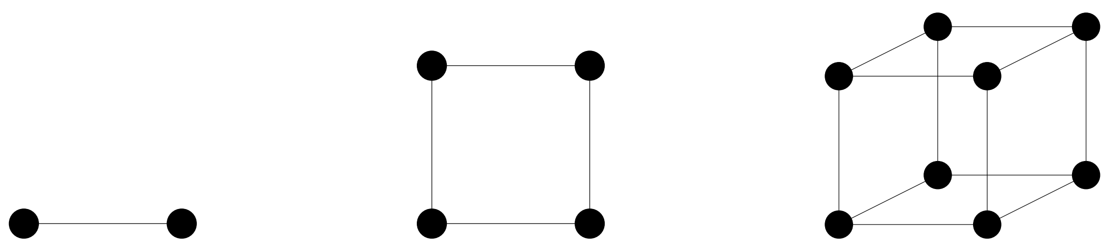

Figure 4: The interval, the square, and the cube. Draw the tesseract as an exercise.

In general we take an n-cube to be the graph $Q^n = I \times I \times \ldots \times I$: the product of the interval with itself n times. The vertices of the n-cube are in bijection with the set of binary strings of length n. Two vertices are adjacent in the cube if and only if the corresponding binary strings differ in exactly one position.

Our last family of examples are the *bipartite* graphs. Here is the definition:

Definition 5.3. A graph G is *bipartite* if we can partition the vertex set $V = A \cup B$ so that $A \cap B = \emptyset$, no edge of G has both endpoints in A and no edge of G has both endpoints in B. The *complete bipartite* graph $K_{m,n}$ has $|A| = m$, has $|B| = n$, and has an edge $\{a, b\}$ for all $a \in A$ and $b \in B$.

So, for example, bipartite graphs never have loops. Note that the graph on the right hand side of Figure 3 is a copy of $K_{3,3}$.

Exercise 5.4. Show that $K_{m,n}$ has exactly $m \cdot n$ edges.

Exercise 5.5. Give a direct proof that the n-cube is a bipartite graph. (Hint: Add up all the ones in the binary string, mod 2.)

We end this section with a nice characterization of bipartite graphs:

Theorem 5.6. *A graph is bipartite if and only if all cycles in the graph have even length.*

Proof. Here is just a hint of the proof. Suppose, to begin with, that G is bipartite. Let $v_0, v_1, \ldots, v_n = v_0$ be a cycle. We want to show that n is even. Suppose that v_0 is in A. Now prove by induction that $v_i \in A$ if and only if i is even. Since $v_n = v_0 \in A$ it will follow that n is even.

Suppose now that all cycles have even length. Fix $v \in V(G)$ to be a basepoint. Define $A \subset V(G)$ to be the set of all those vertices which can be reached by some path of even length, starting at v_0. Define $B \subset V(G)$ to be the set of all those vertices which can be reached by some path of odd length, starting at v_0. We leave it to the reader to show that $A \cup B = V(E)$ and $A \cap B = \emptyset$. We also leave it to the reader to show that there is no edge between vertices of A and there is no edge between vertices of B.

Theorem 4.1 and Lemma 2.2 should be quite useful! $\square$

6 Matrices

To investigate the path structure of a graph there are two kinds of matrices which suggest themselves. The first is the *adjacency matrix*, constructed as follows. Label all of the vertices of G so that $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $A(G)$ be the matrix where the entry a_{ij} equals the number of edges connecting v_i to v_j.

We make several observations about $A(G)$:

- We have $\deg(v_i) = \sum_j a_{ij}$.
- $A(G)$ is non-negative and symmetric.
- $A(G)$ records enough information to recover the graph G.
- The ij^{th} entry of the matrix $A^k(G)$, the k^{th} power of $A(G)$, records the number of walks of length exactly k between v_i and v_j. For example, if G is connected and has an odd cycle then there is a k so that all entries of $A^k(G)$ are positive.

Exercise 6.1. Write down the adjacency matrix for the complete bipartite graph $K_{3,3}$ on vertices $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. What do you notice about $A(K_{3,3})$? Describe the powers of $A(K_{3,3})$.

Suppose that G is any bipartite graph. Show that $A(G)$ is a *block anti-diagonal matrix*. Describe qualititively what the powers of $A(G)$ look like.

We now turn to the *incidence matrix* of G. Label all of the vertices of G so that $V(G) = \{v_1, v_2, \dots, v_n\}$. Label all fo the edges of G so that $E(G) = \{e_1, e_2, \dots, e_m\}$. Let $B(G)$ be the matrix where the entry b_{ij} is a one if v_i is an endpoint of e_j.

We make several observations about $A(G)$:

- We have $\deg(v_i) = \sum_j b_{ij}$ and $2 = \sum_i b_{ij}$.
- $B(G)$ is non-negative, but usually is not square.
- $B(G)$ records enough information to recover the graph G.