Problem 5.1. Solve all problems on the midterm exam. Try to find not only the answer, but also a pretty way to explain the answer.

Problem 5.2 (Updated slightly). A n by m *matrix* is a rectangular array of numbers, with n rows and m columns. For example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

A matrix need not be square:

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Suppose that X is a p by n matrix and Y is a n by m matrix. We define a new p by m matrix $X \cdot Y$ where the ij^{th} entry is the dot product of the i^{th} row of X with the j^{th} column of Y. For example:

$$B \cdot C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

- Is $A \cdot C = C \cdot A$?
- Compute $C^2 = C \cdot C$. Compute $C^3 = C^2 \cdot C$. Compute all powers of C: C^2, C^3, C^4, and so on.
- Can you compute all powers of A? Of B?
- Determine which of A, B, C, D, E, F (as above) can be multiplied. For those which can be multiplied, decide the number of rows and columns of the product.

The entries of a matrix need not be constants. For example the matrices

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

are called the *rotation* matrices. The matrices

$$S_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

are called *shear* matrices. These names come from thinking of R_θ and S_t as functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

- Pick a nice value of θ (say, $\pi/3$) and compute the product of R_θ with various 2 by 1 matrices (ie vectors in $\mathbb{R}^2$).

- Do the same for S_t.
- Compute $R_{-\theta} \cdot R_\theta$ and $S_{-t} \cdot S_t$. Compute $R_{2\pi}$. Can you explain what is going on geometrically?

The most important quantity associated to a two by two matrix is the determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Notice that this is identical to the cross product of the rows of the matrix.

- Compute the determinant.of all of the two by two matrices above (including the powers).
- Find a relationship between $\det(X)$, $\det(Y)$, and $\det(X \cdot Y)$. Check algebraically that the relationship holds for any X and Y. Also try to explain the relation geometrically.

The total derviative of a function is also a matrix: As an exercise, compute the total derivative of the function $p(r, \theta) = (r\cos(\theta), r\sin(\theta))$.