Today's workshop will concentrate on a single object: the solid torus standardly embedded in $\mathbb{R}^3$. Pick two numbers $a > b > 0$. Form in the xz plane the circle of radius b, centered at the point $(a, 0, 0)$. Rotate this about the z axis to obtain a solid of revolution which we will label by T or $T(a, b)$. The boundary, ∂T, is two-dimensional and is called a "torus." Note that T is called a "solid torus."

Problem 8.1. Draw the cross sections of T and ∂T which are parallel to the coordinate planes. What is the difference between those of T and those of ∂T?

Give a sketch of T in the xyz space, with the coordinate axes in the standard position.

Problem 8.2. Using one variable techniques (ie shells) find the volume $V(a, b)$ of $T(a, b)$. (Hint: remember to look for symmetry in the integrand, such as even or odd functions.)

Problem 8.3. Using one variable techniques (ie strips) find the area $S(a, b)$ of $\partial T(a, b)$. Once you have computed $V(a, b)$ and $S(a, b)$ try to rewrite them in a geometrically meaningful way.

$\diamond$ Now that we've recalled some of the difficulties of one-variable calculus, let's find out if our fancy-schmancy three dimensional techniques are any easier.

Problem 8.4. Let Q be the solid in uvw space bounded by the cylinder $u^2 + w^2 = 1$ and the pair of planes $v = 0$ and $v = 2\pi$. Put another way, $Q = \{(u, v, w) \mid u^2 + w^2 \leq 1, v \in [0, 2\pi]\}$. Sketch Q in uvw space, with the coordinate axes in the standard position. Find the volume of Q. Find the surface area of Q.

Now give a nice transformation $\mathbf{r}\colon Q \to T$ from uvw space to xyz space which throws Q onto T. The point $(0, 0, 0)$ should be sent to $(a, 0, 0)$ and the v-axis should be sent to the "core circle" of T. Also, $\mathbf{r}_w$ should be a constant multiple of the vector $\mathbf{k}$. Where are planes, parallel to the uv, vw, and wu-coordinate planes, sent by $\mathbf{r}$?

Problem 8.5. Writing the parameterization $\mathbf{r}$ as $\mathbf{r} = (x, y, z)$ (where each of x, y, and z are functions of the variables u, v, and w) compute all of the partial derivative. Pause to admire.

Problem 8.6. Compute the integral $\iiint_T dV = \iiint_T dx\, dy\, dz$ using the map $\mathbf{r}$ and the change of variables formula.

$\diamond$ Let us think quite generally for a moment: Suppose that F is any nice surface in $\mathbb{R}^3$. Suppose that $\mathbf{r}\colon D \to F$ is a map from the $\phi\theta$ plane to xyz space throwing a region D onto the surface F. Then the book claims that the surface area of F is:

$$\iint_F dF = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi\, d\theta.$$

Problem 8.7. Choose a nice map $\mathbf{s}(\phi, \theta)$ from the $\phi\theta$ plane to the uvw space which throws the square $[0, 2\pi]\times[0, 2\pi]$ onto the cylindrical part of ∂Q. (Hint: take $v(\phi, \theta) = \phi$.) Compute a totally explicit expression for $\mathbf{R}(\phi, \theta) = \mathbf{r}(\mathbf{s}(\phi, \theta))$. Use this to again find $S(a, b)$, the area of ∂T, using the formula above.