Please write in complete English sentences. Helpful figures are always welcome.

Todays workshop is a review of all of the "derivatives" we have seen in class. Recall that ∇ is the "del" operator. We have $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ in dimension two and $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ in dimension three. Now let us list all the derivatives in dimension two:

- If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a scalar function then we defined $\nabla f = \langle f_x, f_y \rangle$, the *gradient* of f.
- If $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, say $F = \langle P, Q \rangle$, then we defined $\nabla \times F = Q_x - P_y$, the *curl* of F. This is sometimes written curl F.

Here are all of the derivatives in dimension three:

- As above, if $f\colon \mathbb{R}^3 \to \mathbb{R}$ then $\nabla f = \langle f_x, f_y, f_z \rangle$ is the gradient of f.
- Similar to the above, if $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, say $F = \langle P, Q, R \rangle$, then we define
$$\nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$
This is also called the curl of F and sometimes written curl F.
- Also, if $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then we define $\nabla \cdot F = P_x + Q_y + R_z$, the *divergence* of F. This is sometimes written div F.

Problem 10.1. Check that, in dimension two, $\nabla \times (\nabla f) = 0$ for any function f. That is, the curl of the gradient is always zero. (If you have trouble doing this, check a few examples like $f(x, y) = x^2 + y^2$ or $g(x, y) = e^x \cos(y)$. Clairaut's Theorem on page 916 may be helpful.)

Problem 10.2. Check that this also holds in dimension three $\nabla \times (\nabla f) = 0$. (Again, do a few examples first: like $f(x, y, z) = xyz$ or $g(x, y, z) = ye^{x^2+z}$.) The fundamental theorem for line integrals explains why this works – gradient vector fields are always conservative.

Problem 10.3. Also in dimension three, check that

$$\nabla \cdot (\nabla \times F) = 0.$$

The above three exercises can be summarized by saying "taking two derivatives yields zero." This is very similar to the fact, observed in class, that $\partial\partial D = \emptyset$: for any domain D, the boundary of the boundary of D is empty.

Problem 10.4. Recall that the one variable product rule says $(fg)' = f'g + fg'$. Similar rules hold in higher dimensions. For example: suppose that f and g are both functions on $\mathbb{R}^3$. Find a formula for the gradient of the product fg in terms of ∇f and ∇g.

Problem 10.5. If f is a function and F is a vector field, both in the same dimension, then we can define a new vector field by scaling: $G = fF$. Derive formulas for $\nabla \times G$ (in dimensions two and three) and for $\nabla \cdot G$ in terms of f, F, and their derivatives.

Problem 10.6. (Hard.) Here is a final product rule problem: suppose that F and G are both vector fields. Find expressions for the derivatives of $F \cdot G$ and $F \times G$ in dimensions two and three. (Compare to problem 20 on page 1136 of the book.) An important special case is the gradient of $|F|$.