Lecture notes of Simon Thomas. Transcribed by Saul Schleimer.

1 Introduction

In this course we will be studying set theory:

- Set Theory as a foundation for all mathematics
- Set Theory as a subject in its own right.

Set Theory as Foundation

At the end of the 19^{th} Century, mathematicians became concerned that they didn't fully understand the nature of the basic mathematical notions. For example:

- What do we mean by a function $f : \mathbb{R} \to \mathbb{R}$? How general should we take this notion?
- We are have an intutive understanding of the set $\mathbb{N}$ of natural numbers. Using this, we can also understand the integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$. But what about the reals $\mathbb{R}$?

What caused this concern? The discovery of certain contradictions...

Russell's Paradox

Consider the set $A = \{x | x \notin x\}$. Is A a member of itself? First suppose that $A \notin A$. Then A satisfies the defining property of elements of A and so $A \in A$, which is a contradiction. Next suppose that $A \in A$. Then A does not satisfy the defining property of elements of A; ie $A \notin A$, which is a contradiction.

So we have to provide a more careful axiomatic foundation of set theory, which doesn't lead to contradictions. In particular, this more careful approach should explain the flaw in Russell's Paradox.

Suppose that we have accomplished this. Then it is necessary to do the same for even more complicated theories:

- Number Theory $\mathbb{N}$
- Analysis $\mathbb{R}$
- *etc.*

Thus it seems necessary to successively develop safe foundations for *every* area of mathematics.

Another approach... we shall show that all of mathematics can be reduced to / defined in terms of set theory. In other words, every mathematical notion can be defined in terms of the set membership relation $\in$. So it is enough to provide a safe foundation for set theory.

Set Theory in its own right

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2 Basic Set Theory

Axiom 2.1 (Extensionality). *If the sets A, B have exactly the same members, then $A = B$.*

Definition 2.2. The set A is a *subset* of the set B, written $A \subseteq B$, iff every member of A is also a member of B.

Proposition 2.3. *Let A, B be sets. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

Proof. Let x be arbitrary. Suppose that $x \in A$. Since $A \subseteq B$, it follows that $x \in B$. Now suppose that $x \in B$. Since $B \subseteq A$, it follows that $x \in A$. Thus $x \in A$ iff $x \in B$. Hence $A = B$. □

Axiom 2.4 (Empty Set). *There exists a set whch has no members.*

By the Extensionality Axiom, there exists a unique such set which we denote by $\emptyset$, the *empty set*.

Axiom 2.5 (Union, Preliminary Form). *For any sets A, B there exists a set C such that for all x,*

$$x \in C \quad \textit{iff} \quad x \in A \quad \textit{or} \quad x \in B$$

Once again, there exists a unique such set which we denote by $A \cup B$, the *union* of A and B.

Proposition 2.6. *If A, B are sets, then $A \cup B = B \cup A$.*

Proof. Let x be arbitrary. Then $x \in A \cup B$
iff $x \in A$ or $x \in B$
iff $x \in B$ or $x \in A$
iff $x \in B \cup A$.
Hence $A \cup B = B \cup A$. □

Axiom 2.7 (Subset). *Suppose that A is a set and $\varphi(x)$ is a property. The there exists a set B such that for all x,*

$$x \in B \text{ iff } x \in A \text{ and } \varphi(x) \text{ holds.}$$

Once again, there is a unique such set which we denote by $\{x \in A \mid \varphi(x)\}$.

Definition 2.8. Let A, B be sets. Then

$$A \cap B = \{x \in A \mid x \in B\}$$

and

$$A \smallsetminus B = \{x \in A \mid x \notin B\}.$$

Remark 2.9. Applying the Subset Axiom, if A and B are any sets, then $A \cap B$ and $A \smallsetminus B$ are also sets.

In other words

$$x \in A \cup B \quad \text{iff} \quad x \in A \quad \text{or} \quad x \in B$$

$$x \in A \cap B \quad \text{iff} \quad x \in A \quad \text{and} \quad x \in B$$

$$x \in A \smallsetminus B \quad \text{iff} \quad x \in A \quad \text{and} \quad x \notin B$$

Exercise 2.10. If A, B are sets, then $A \cap B = B \cap A$.

Proposition 2.11. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Let x be arbitrary. Then $x \in A \cup (B \cup C)$
iff $x \in A$ or $x \in B \cup C$
iff $x \in A$ or $x \in B$ or $x \in C$
iff $x \in A \cup B$ or $x \in C$
iff $x \in (A \cup B) \cup C$
Thus $A \cup (B \cup C) = (A \cup B) \cup C$. □

Exercise 2.12. $A \cap (B \cap C) = (A \cap B) \cap C$

Proposition 2.13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. Let x be arbitrary. Then $x \in A \cap (B \cup C)$
iff $x \in A$ and $x \in B \cup C$
iff $x \in A$ and ($x \in B$ or $x \in C$)
iff ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)
iff $x \in A \cap B$ or $x \in A \cap C$
iff $x \in (A \cap B) \cup (A \cap C)$
Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Exercise 2.14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proposition 2.15. $C \smallsetminus (A \cup B) = (C \smallsetminus A) \cap (C \smallsetminus B)$

Proof. Let x be arbitrary. Then $x \in C \smallsetminus (A \cup B)$
iff $x \in C$ and $x \notin A \cup B$
iff $x \in C$ and not ($x \in A$ or $x \in B$)
iff $x \in C$ and ($x \notin A$ and $x \notin B$)
iff $x \in C$ and $x \notin A$ and $x \notin B$
iff ($x \in C$ and $x \notin A$) and ($x \in C$ and $x \notin B$)
iff ($x \in C \smallsetminus A$) and ($x \in C \smallsetminus B$)
iff $x \in (C \smallsetminus A) \cap (C \smallsetminus B)$
Thus $C \smallsetminus (A \cup B) = (C \smallsetminus A) \cap (C \smallsetminus B)$. □

Exercise 2.16. $C \smallsetminus (A \cap B) = (C \smallsetminus A) \cup (C \smallsetminus B)$.

Next we want an axiom which states the existence of unions of possibly infinite collections of sets.

Axiom 2.17 (Union). *For any set A, there exists a set B such that for all x,*

$$x \in B \quad \textit{iff} \quad (\exists a \in A)\ x \in a.$$

Once again, there is a unique such set which we denote by $\bigcup A$.

Example 2.18. Suppose that

$$A = \{\{1, 2, 3\}, \{5, 7\}, \{3, 8\}\}.$$

Then

$$\bigcup A = \{1, 2, 3, 5, 7, 8\}.$$

In order to prove that the union of two sets exist, we need to add the following axiom.

Axiom 2.19 (Pairing). *For any two sets A, B, there exists a set C such that for all x,*

$$x \in C \quad \textit{iff} \quad x = A \quad \textit{or} \quad x = B.$$

Once again, there is a unique such set which we denote by $\{A, B\}$.

Remark 2.20. Let A, B be any sets. By Pairing, there exists a set $C = \{A, B\}$. By Unions, there exists a set $\bigcup C = A \cup B$. Thus we can delete the preliminary form of the Union Axiom from our list of axioms.

Definition 2.21. For any *non-empty* set A, we define the intersection $\bigcap A$ by for all x,

$$x \in \bigcap A \quad \text{iff} \quad (\forall a \in A)\ x \in a.$$

Example 2.22. For each $n \geq 1$, let $I_n = (-1/n, 1/n) \subseteq \mathbb{R}$. Let

$$A = \{I_1, I_2, I_3, \ldots, I_n, \ldots\}$$

Then $\bigcap A = \{0\}$.

Theorem 2.23. *For any nonempty set A, then exists a set B such that for all x,*

$$x \in B \quad \textit{iff} \quad (\forall a \in A)\ x \in a.$$

Proof. Choose some fixed $c \in A$ and let $\varphi(x)$ be the property such that

$$\varphi(x) \text{ holds iff } x \in a \text{ for all } a \in A.$$

By the Subset Axiom, there exists a set B such that for all x,

$$x \in B \text{ iff } x \in c \text{ and } \varphi(x) \text{ holds.}$$

Clearly B satisfies our requirements. □

The distributive and deMorgan laws also hold for arbitrary intersections and unions.

Theorem 2.24. *Let $\mathcal{B}$ be a nonempty set of sets and let A be any set.*

(a). $A \cup \bigcap \mathcal{B} = \bigcap \{A \cup B \mid B \in \mathcal{B}\}$

(b). $A \cap \bigcup \mathcal{B} = \bigcup \{A \cap B \mid B \in \mathcal{B}\}$

(c). $A \smallsetminus \bigcup \mathcal{B} = \bigcap \{A \smallsetminus B \mid B \in \mathcal{B}\}$

(d). $A \smallsetminus \bigcap \mathcal{B} = \bigcup \{A \smallsetminus B \mid B \in \mathcal{B}\}$

Proof. (a) Let a be arbitrary. Then $x \in A \cup \bigcap \mathcal{B}$
iff $x \in A$ or (for all $B \in \mathcal{B}, x \in B$)
iff for all $B \in \mathcal{B}, x \in A \cup B$
iff $x \in \bigcap \{A \cup B \mid B \in \mathcal{B}\}$
(c) $x \in A \smallsetminus \bigcup \mathcal{B}$ iff $x \in A$ and $x \notin \bigcup \mathcal{B}$
iff $x \in A$ and not (there exists $B \in \mathcal{B}, x \in B$)
iff $x \in A$ and (for all $B \in \mathcal{B}, x \notin B$)
iff for all $B \in \mathcal{B}, x \in A \smallsetminus B$
iff $x \in \bigcap \{A \smallsetminus B \mid B \in \mathcal{B}\}$. □

Exercise 2.25. Prove (b) and (d).

In the next section, we will also need the following axiom.

Axiom 2.26 (Power Set). *For any set A, there exists a set B such that for all x,*

$$x \in B \quad \textit{iff} \quad x \subseteq A.$$

Once again, there is a unique such set which we denote by $\mathcal{P}A$, the *powerset* of A. (We sometimes write $\mathcal{P}(A)$.)

Example 2.27. $\mathcal{P}\{0,1,2\} = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$

In general, if A is a finite set with $|A| = n$, then $|\mathcal{P}A| = 2^n$.

Question. Which sets can we prove exist using our current set of axioms?

Remark. Clearly we can show that $\emptyset$ exists. If A is a set, then we can prove that $A \cup \{A\}$ exists. Hence we can prove inductively that $n \in \mathbb{N}$ exists.