3 Relations and functions

Preliminary Definition of a Function

Let A, B be sets. Then a function $f\colon A \to B$ is a rule which assigns to each element $a \in A$ a unique element $f(a) \in B$.

We want to reduce this notion to set theory. So a function should be a certain kind of set. Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

Eventually we shall define

$$g = \{\langle x, x^2 \rangle \mid x \in \mathbb{R}\}$$

In general, a function $f\colon A \to B$ will be defined to be a certain set of ordered pairs $\langle a, b \rangle$, where $a \in A$ and $b \in B$.

What is an *ordered pair*? An object such that $\langle a, b \rangle = \langle c, d \rangle$ iff $a = c$ and $b = d$.

In particular, $\langle a, b \rangle \neq \{a, b\}$. Order counts in the former but not in the latter.

Definition 3.1. We define $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$.

We must check that this definition works.

Theorem 3.2. *$\langle u, v \rangle = \langle x, y \rangle$ iff $u = x$ and $v = y$.*

Proof. $\Leftarrow$ If $u = x$ and $v = y$, then clearly

$$\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$$

ie $\langle u, v \rangle = \langle x, y \rangle$.
$\Rightarrow$ Suppose that $\langle u, v \rangle = \langle x, y \rangle$, *ie* $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$.

Case 1 Suppose that $u = v$. Then

$$\begin{aligned} \{\{u\}, \{u, v\}\} &= \{\{u\}, \{u, u\}\} \\ &= \{\{u\}, \{u\}\} \\ &= \{\{u\}\} \end{aligned}$$

Since

$$\{\{x\}, \{x, y\}\} = \{\{u\}\}$$

it follows that

$$\{x\} = \{u\} \quad \text{and} \quad \{x, y\} = \{u\}$$

Thus $x = y = u$. Hence $u = x$ and $v = y$.

Case 2 Suppose that $x = y$. By a similar argument, we find that $x = y = u = v$. Hence $u = x$ and $v = y$.

Case 3 Suppose that $u \neq v$ and $x \neq y$. Since

$$\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$$

we must have that (a) $\{u\} = \{x\}$ or (b) $\{u\} = \{x, y\}$. Clearly (b) is impossible, since $\{u\}$ contains one element and $\{x, y\}$ contains two elements. Thus $\{u\} = \{x\}$ and so $u = x$. Also, we must have that (c) $\{u, v\} = \{x\}$ or (d) $\{u, v\} = \{x, y\}$. Again (c) is clearly impossible and so

$$\{u, v\} = \{x, y\} = \{u, y\}$$

It follows that $v = y$. □

Question. Suppose that x, y are sets. Do our current axioms prove that $\langle x, y\rangle$ is a set?

Answer. Yes! Suppose that x, y are sets. Applying the Pairing Axiom, we see that $\{x, y\}$ and $\{x, x\} = \{x\}$ are both sets. Applying the Pairing Axiom once more,

$$\{\{x\}, \{x, y\}\}$$

is also a set.

Definition 3.3. Let A, B be sets. Then their *Cartesian product* is defined to be

$$A \times B = \{\langle x, y\rangle \mid x \in A \quad \text{and} \quad y \in B\}.$$

Question. Suppose that A, B are sets. Do our current axioms prove that $A \times B$ is a set?

Answer. Yes! But this requires a bit more effort...

Lemma 3.4. *Let C be a set. If $x, y \in C$, then $\langle x, y\rangle \in \mathcal{P}\mathcal{P}C$.*

Proof. Suppose that $x, y \in C$. Then $\{x\} \subseteq C$ and $\{x, y\} \subseteq C$. Thus $\{x\}, \{x, y\} \in \mathcal{P}C$. Hence $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$ and so

$$\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$$ □

Theorem 3.5. *Suppose that A, B are sets. Then there exists a set D such that for all x,*

$$x \in D \text{ iff there exists } a \in A \text{ and } b \in B \text{ such that } x = \langle a, b\rangle.$$

In other words, the Cartesian product of A and B is a set.

Proof. Let A, B be sets. By Pairing, $\{A, B\}$ is a set. By Union, $\bigcup\{A, B\} = A \cup B$ is a set. Applying Powerset twice, we see that $\mathcal{P}\mathcal{P}(A \cup B)$ is a set. Futhermore, by the Lemma, $\langle a, b\rangle \in \mathcal{P}\mathcal{P}(A \cup B)$ for all $a \in A$ and $b \in B$. By Subset, there exists a set D such that

$$x \in D \text{ iff } x \in \mathcal{P}\mathcal{P}(A \cup B) \text{ and } x = \langle a, b\rangle \text{ for some } a \in A \text{ and } b \in B.$$

Cleaarly D satisfies our requirements. □

Definition 3.6. Let A, B be sets. Then f is a *function* from A to B, written $f\colon A \to B$, iff

- $f \subseteq A \times B$; and
- for each $a \in A$, there exists a unique $b \in B$ such that $\langle a, b\rangle \in f$. We denote this unique element b by $f(b)$.

Definition 3.7. Let A, B be sets. Then

$$B^A = \{f \mid f\colon A \to B\}.$$

It is easily seen that our current axioms imply that B^A is a set. To see this, note that if $f\colon A \to B$, then $f \subseteq A \times B$ and so $f \in \mathcal{P}(A \times B)$. By Subset,

$$B^A = \{f \in \mathcal{P}(A \times B) \mid f \text{ is a function from } A \text{ to } B\}$$

is a set.

We shall develop the basic theory of functions in a little while. First we want to introduce the more general notion of a relation. On second thoughts... we'll develop the basic theory of functions.

Definition 3.8. A function $f\colon A \to B$ is *one-to-one* / an *injection* iff for all $a_1, a_2 \in A$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.

Example 3.9.

- $f\colon \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$, is an injection.
- $g\colon \mathbb{Z} \to \mathbb{Z}$, $g(z) = z^2$, isn't an injection, since $g(1) = 1 = g(-1)$.

Definition 3.10. A function $f\colon A \to B$ is *onto* / a *surjection* iff for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Example 3.11.

- $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(z) = z + 1$, is a surjection.
- $g\colon \mathbb{N} \to \mathbb{N}$, $g(n) = n + 1$, isn't a surjection, since there doesn't exist $n \in \mathbb{N}$ such that $g(n) = 0$.

Definition 3.12. Suppose that $f\colon A \to B$ and $C \subseteq A$. Then

$$f[C] = \{f(c) \mid c \in C\}.$$

Thus $f\colon A \to B$ is a surjection iff $f[A] = B$.

Definition 3.13. Suppose that $f\colon A \to B$ and $g\colon B \to C$. Then their *composition* is the function $g \circ f\colon A \to C$ defined by

$$(g \circ f)(a) = g(f(a)).$$

Example 3.14. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = x^2 + 1$ and $g(x) = \sin(x)$. Then

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 + 1) \\ &= \sin(x^2 + 1)\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sin(x)) \\ &= \sin^2(x) + 1\end{aligned}$$

Proposition 3.15. *If $f\colon A \to B$ and $g\colon B \to C$ are surjections, then $g \circ f\colon A \to C$ is also a surjection.*

Proof. Let $c \in C$. Since g is a surjection, there exists $b \in B$ such that $g(b) = c$. Since f is a surjection, there exists $a \in A$ such that $f(a) = b$. Hence

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c \quad \square\end{aligned}$$

Exercise 3.16. If $f\colon A \to B$ and $g\colon B \to C$ are injections, then $g \circ f$ is also an injection.

Definition 3.17. A function $f\colon A \to B$ is a *bijection* iff f is both an injection and a surjection.

Remark 3.18. Thus $f\colon A \to B$ is a bijection iff for each $b \in B$ there is a unique $a \in A$ such that $f(a) = b$.

Example 3.19. For each set A, we define the *identity function* on A

$$I_A\colon A \to A$$

by $I_A(a) = a$. Clearly I_A is a bijection.

Definition 3.20. Suppose that $f\colon A \to B$ is a bijection. Then we can define the *inverse function* $f^{-1}\colon B \to A$ by

$$f^{-1}(b) = \text{the unique } a \in A \text{ such that } f(a) = b.$$

Remark 3.21. Thus we have that $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$. Also notice that f^{-1} is a bijection and that $(f^{-1})^{-1} = f$. Also notice that

$$f^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in f\}.$$

Proposition 3.22. *If $f\colon A \to B$ and $g\colon B \to C$ are bijections, then $g \circ f\colon A \to C$ is also a bijection.*

Proof. Immediate consequence of the corresponding results for injections and surjections. □

Theorem 3.23. *If $f\colon A \to B$ and $g\colon B \to C$ are bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Example 3.24. Consider $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x + 2$ *etc.*

Proof of Theorem. Let $c \in C$. Let $b = g^{-1}(c)$, so that $g(b) = c$. Let $a = f^{-1}(b)$, so that $f(a) = b$. Then

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c \end{aligned}$$

Hence $(g \circ f)^{-1}(c) = a$. Also

$$\begin{aligned} (f^{-1} \circ g^{-1})(c) &= f^{-1}(g^{-1}(c)) \\ &= f^{-1}(b) \\ &= a \end{aligned}$$

Hence $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$. □