11 Construction of $\mathbb{R}$

Theorem 11.1. *There does not exist $q \in \mathbb{Q}$ such that $q^2 = 2$.*

Proof. Suppose that $q \in \mathbb{Q}$ satisfies $q^2 = 2$. Then we can suppose that $q > 0$ and express $q = a/b$, where $a, b \in \omega$ are relatively prime. Since

$$\frac{a^2}{b^2} = 2$$

we have that

$$a^2 = 2b^2.$$

It follows that $2|a$; say $a = 2c$. Hence

$$\begin{aligned} 4c^2 &= 2b^2 \\ 2c^2 &= 2b^2 \end{aligned}$$

This means that $2|b$, which contradicts the assumption that a and b are relatively prime. □

Theorem 11.2. *There exists $r \in \mathbb{R}$ such that $r^2 = 2$.*

Proof. Consider the continuous function $f\colon [1, 2] \to \mathbb{R}$, defined by $f(x) = x^2$. Then $f(1) = 1$ and $f(2) = 4$. By the Intermediate Value Theorem, there exists $r \in [1, 2]$ such that $r^2 = 2$. □

Why is the Intermediate Value Theorem true? Intuitively because $\mathbb{R}$ "has no holes"... More precisely...

Definition 11.3. Suppose that $A \subseteq \mathbb{R}$.

- $r \in \mathbb{R}$ is an *upper bound* of A iff $a \leq r$ for all $a \in A$.
- A is *bounded above* iff there exists an upper bound for A.
- $r \in \mathbb{R}$ is a *least upper bound* of A iff the following conditions hold:
  - r is an upper bound of A.
  - If s is an upper bound of A, then $r \leq s$.

Axiom 11.4 (Completeness). *If $A \subseteq \mathbb{R}$ is nonempty and bounded above, then there exists a least upper bound of A.*

Deepish Fact Completeness Axiom $\Rightarrow$ Intermediate Value Theorem.

Remark 11.5. The analogue of the Completeness Axiom fails for $\mathbb{Q}$. *eg* consider

$$C = \{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\}.$$

Then C is a nonempty subset of $\mathbb{Q}$ which is bounded above but has no least upper bound. The least upper bound "should be" $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. Our construction of $\mathbb{R}$ will be such that

$$\{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\} = \sqrt{2}.$$

Target We now want to extend $\mathbb{Q}$ to a larger set of "numbers" $\mathbb{R}$ so that the Completeness Axiom holds.

Definition 11.6. A *Dedekind cut* is a subset $x \subseteq \mathbb{Q}$ such that:

- $\emptyset \neq x \neq \mathbb{Q}$.
- x is "downward closed"; *ie* if $q \in x$ and $r < q$, then $r \in x$.
- x has *no* largest element.

Example 11.7.

- For each $q \in QQ$, the set

$$E(q) = \{r \in \mathbb{Q} \mid r < q\}$$

is a Dedakind cut.

-

$$\{r \in \mathbb{Q} \mid r \leq 0 \text{ or } r^2 < 2\}$$

is a Dedekink cut.

Definition 11.8. $\mathbb{R}$ is the set of Dedekind cuts.

Definition 11.9. We define the binary relation $<_{\mathbb{R}}$ on $\mathbb{R}$ by

$$x <_{\mathbb{R}} y \quad \text{iff} \quad x \subsetneq y.$$

Theorem 11.10. *$<_{\mathbb{R}}$ is a linear order on $\mathbb{R}$.*

Proof. Suppose that $x <_{\mathbb{R}} y$ and $y <_{\mathbb{R}} z$. Then $x \subsetneq y$ and $y \subsetneq z$. It follows that $x \subsetneq z$ and so $x <_{\mathbb{R}} z$. Thus $<_{\mathbb{R}}$ is transitive.

Now we shall prove that $<_{\mathbb{R}}$ satisfies trichotomy. Let $x, y \in \mathbb{R}$. Clearly at most one of the following holds:

$$x \subsetneq y, \quad x = y, \quad y \subsetneq x.$$

Suppose that the first two fail; *ie* $x \nsubseteq y$. We must prove that $y \subsetneq x$.

Since $x \nsubseteq y$, there exists $r \in x \setminus y$. Consider any $q \in y$. If $r \leq q$, then $r \in y$ since y is downward closed, which is a contradiction. Thus $q < r$. Since x is downward closed, $q \in x$. Thus $y \subsetneq x$. □

Theorem 11.11. *If $A \subseteq \mathbb{R}$ is nonempty and bounded above, then there exists a least upper bound for A.*

Proof. We shall prove that $\bigcup A$ is the least upper bound of A.

Claim. $\bigcup A \in \mathbb{R}$.

Proof. Let $x \in A$. Then $x \subseteq \bigcup A$ and so $\bigcup A \neq \emptyset$. Now let $z \in \mathbb{R}$ be an upper bound of A. Then $x \subseteq z$ for all $x \in A$ and so $\bigcup A \subseteq z$. Since $z \neq \mathbb{Q}$, it follows that $\bigcup A \neq \mathbb{Q}$.

Now suppose that $q \in \bigcup A$ and $r < q$. Then there exists $x \in A$ such that $q \in x$ and clearly $r \in x$. Thus $r \in bigcupA$. So $\bigcup A$ is downward closed.

Finally suppose that $q \in \bigcup A$ is the largest element. Then there exists $x \in A$ such that $q \in x$ and clearly q is the largest element of xm which is a contradiction. Hence $\bigcup A$ has *no* largest element. □

Since $x \subseteq \bigcup A$ for all $x \in A$, it follows that $\bigcup A$ is an upper bound for A. By the above argument, if $z \in \mathbb{R}$ is any upper bound of A, then $\bigcup A \subseteq z$. Hence $\bigcup A$ is the least upper bound of A. □

We next want to define an additive operation on $\mathbb{R}$. Note that

$$\begin{aligned} E(1) &= \{r \in \mathbb{Q} \mid r < 1\} \\ E(2) &= \{s \in \mathbb{Q} \mid s < 2\} \end{aligned}$$

and hence

$$E(3) = \{t \in \mathbb{Q} \mid t < 3\} = \{r + s \mid r \in E(1) \text{ and } s \in E(2)\}.$$

This suggests we make the following definition.

Definition 11.12. We define the binary operation $+_{\mathbb{R}}$ on $\mathbb{R}$ by

$$x +_{\mathbb{R}} y = \{q + r \mid q \in x \text{ and } r \in y\}$$

.

Lemma 11.13. *If $x, y \in \mathbb{R}$, then $x +_{\mathbb{R}} y \in \mathbb{R}$.*

Proof. Since $x, y \neq \emptyset$, it follows that $x +_{\mathbb{R}} y \neq \emptyset$. To see that $x +_{\mathbb{R}} y \neq \mathbb{Q}$, choose some $q' \in \mathbb{Q} \smallsetminus x$ and $r' \in \mathbb{Q} \smallsetminus y$. Then we have that:

- $q < q'$ for all $q \in x$
- $r < r'$ for all $r \in y$

It follows that if $q \in x$ and $r \in y$, then

$$q + r < q' + r'$$

and so $q' + r' \notin x +_{\mathbb{R}} y$. Thus $x +_{\mathbb{R}} y \neq \mathbb{Q}$.

To see that $x +_{\mathbb{R}} y$ is downward closed, suppose that $t \in x +_{\mathbb{R}} y$ and $p < t$. Then there exist $q \in x$ and $r \in y$ such that $t = q + r$. Since

$$p < q + r$$

it follows that $p - q < r$ and so $p - q \in y$. Hence

$$p = q + (p - q) \in x +_{\mathbb{R}} y.$$

Finally we check that $x +_{\mathbb{R}} y$ has no largest element. Let $t \in x +_{\mathbb{R}} y$ be any element. Then there exists $q \in x$ and $r \in y$ such that $t = q + r$. Since x has no largest element, there exists $q < q' \in x$ and so

$$t = q + r < q' + r \in x +_{\mathbb{R}} y.$$

□

Theorem 11.14. *For all $x, y, z \in \mathbb{R}$, we have that*

$$x +_{\mathbb{R}} y = y +_{\mathbb{R}} x$$

$$(x +_{\mathbb{R}} y) +_{\mathbb{R}} z = x +_{\mathbb{R}} (y +_{\mathbb{R}} z).$$

Proof. We just check the first equality.

$$\begin{aligned} x +_{\mathbb{R}} y &= \{q + r \mid q \in x, r \in y\} \\ &= \{r + q \mid r \in y, q \in x\} \\ &= y +_{\mathbb{R}} x \end{aligned}$$

□

Definition 11.15.

$$0_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 0\}$$

Theorem 11.16.

(a) $0_{\mathbb{R}} \in \mathbb{R}$.

(b) For all $x \in \mathbb{R}$, we have $x +_{\mathbb{R}} 0_{\mathbb{R}} = x$.

Proof. (a) is clear! To prove (b) we must prove that

$$\{r + s \mid r \in x \text{ and } s < 0\} = x.$$

Claim. $\{r + s \mid r \in x \text{ and } s < 0\} \subseteq x$.

Proof. Suppose that $r \in x$ and $s < 0$. Then $r + s < r$. Since x is downward closed, $r + s \in x$. □

Claim. $x \subseteq \{r + s \mid r \in x \text{ and } s < 0\}$.

Proof. Let $p \in x$. Since x has no largest element, there exists $p < r \in x$. Thus $p - r < 0$ and

$$p = r + (p - r).$$

□

□

Problem 11.17. How should we define $-\sqrt{2}$? Recall that

$$\sqrt{2} = \{q \in \mathbb{Q} \mid q < 0 \text{ or } q^2 < 2\}.$$

We cannot define

$$-\sqrt{2} = \{q \in \mathbb{Q} \mid -q \in \sqrt{2}\}$$

because this is not downward closed. How about

$$-\sqrt{2} = \{q \in \mathbb{Q} \mid -q \notin \sqrt{2}\}$$

First Candidate: $-x = \{r \in \mathbb{Q} \mid -r \notin x\}$

Fatal flaw! Consider $E(1) = \{q \in \mathbb{Q} \mid q < 1\}$. Then

$$\begin{aligned} -E(1) &= \{r \in \mathbb{Q} \mid -r \not< 1\} \\ &= \{r \in \mathbb{Q} \mid r \leq -1\} \notin \mathbb{R} \end{aligned}$$

since it has a greatest element.

Definition 11.18. For any $x \in \mathbb{R}$, we define

$$-x = \{r \in \mathbb{Q} \mid (\exists s > r) \ -s \notin x\}.$$

Example 11.19.

$$-E(1) = \{r \in \mathbb{Q} \mid r < -1\} \in \mathbb{R}.$$

Theorem 11.20. *For every $x \in \mathbb{R}$, we have:*

(a) $-x \in \mathbb{R}$

(b) $x +_{\mathbb{R}} (-x) = 0_{\mathbb{R}}$

Proof. To see that $-x \neq \emptyset$, choose some $t \notin x$ and let $r = -t - 1$. Then $-(-t) \notin x$ and $r < -t$. Hence $r \in -x$.

To see that $x \neq \mathbb{Q}$, choose any $p \in x$.

Claim. $-p \notin -x$

Proof. If $s > -p$, then $-s < p$ and so $-s \in x$. □

To see that $-x$ is downward closed, suppose that $r \in -x$ and $p < r$. Then there exists $s > r$ such that $-s \notin x$. But clearly $s > p$ and so $p \in -x$.

To see that $-x$ has no largest element, suppose that $r \in -x$. Then there exists $s > r$ such that $-s \notin x$. Since $\mathbb{Q}$ is a dense linear order, there exists $p \in \mathbb{Q}$ such that $s > p > r$ and clearly $p \in -x$.

(b) We must prove that

$$\{q + r \mid q \in x \text{ and } (\exists s > r)\ -s \notin x\} = 0_{\mathbb{R}}.$$

Claim. $\{q + r \mid q \in x \text{ and } (\exists s > r)\ -s \notin x\} \subseteq 0_{\mathbb{R}}$.

Proof. Suppose $q+r$ lies in the LHS set. Since $q \in x$ and $-s \notin x$, we have $q < -s$. Thus

$$q + r < -s + s = 0$$

and so $q + r \in 0_{\mathbb{R}}$. □

Claim. $0_{\mathbb{R}} \subseteq \{q + r \mid q \in x \text{ and } (\exists s > r)\ -s \notin x\}$.

Proof. Suppose that $p \in 0_{\mathbb{R}}$. Then $p < 0$ and so $-p > 0$. It follows that $-p/2 > 0$. This implies that there exists $q \in x$ such that $q + (-p/2) \notin x$. [Why? If not, choose any $t \in x$. Then we see inductively that $t + n(-p/2) \in x$ for all $n \in \omega$. But this means that x contains arbitrarily large rational numbers and so $x = \mathbb{Q}$, which is a contradiction.] Let $s = (p/2) - q$, so that $s \notin x$. Since $-p > -p/2 > 0$, we have that

$$p - q < (p/2) - q = s$$

and so $p - q \in -x$. Also, $q \in x$ and

$$p = q + (p - q).$$

□

□

Now we want to define a multiplication operation on $\mathbb{R}$. Note that

$$E(2) = \{r \in \mathbb{Q} \mid r < 2\}$$

$$E(3) = \{s \in \mathbb{Q} \mid s < 3\}$$

and

$$E(6) = \{t \in \mathbb{Q} \mid t < 6\}$$

First guess

$$E(6) \stackrel{?}{=} \{rs \mid r < 2 \text{ and } s < 3\} \ni 10 = (-2)(-5)$$

Wrong!

$$E(6)\{rs \mid 0 \leq r < 2 \text{ and } 0 \leq s < 3\} \cup 0_{\mathbb{R}}$$

Also note that

$$E(-2) \cdot_{\mathbb{R}} E(-3) = E(2) \cdot_{\mathbb{R}} E(3)$$

and

$$E(-2) \cdot_{\mathbb{R}} E(3) = E(2) \cdot_{\mathbb{R}} E(-3)$$

Definition 11.21. For each $x \in \mathbb{R}$, we define

$$\begin{aligned} |x| &= \max\{x, -x\} \\ &= x \cup (-x) \end{aligned}$$

Exercise 11.22. For all $x \in \mathbb{R}$, $|x| \geq 0_{\mathbb{R}}$.

Definition 11.23. We define the binary operation $\cdot_{\mathbb{R}}$ on $\mathbb{R}$ as follows.

(a) If $x, y \geq 0$, then

$$x \cdot_{\mathbb{R}} y = \{rs \mid 0 \leq r \in x \text{ and } 0 \leq s \in y\} \cup 0_{\mathbb{R}}.$$

(b) If $x, y < 0_{\mathbb{R}}$, then

$$x \cdot_{\mathbb{R}} y = |x| \cdot_{\mathbb{R}} |y|.$$

(c) If exactly one of x, y is negative, then

$$x \cdot_{\mathbb{R}} y = -(|x| \cdot_{\mathbb{R}} |y|).$$

It can be shown (with some difficulty!) that $\cdot_{\mathbb{R}}$ has the usual properties.

Definition 11.24. We define the function $E\colon \mathbb{Q} \to \mathbb{R}$ by

$$E(q) = \{r \in \mathbb{Q} \mid r < q\}.$$

Then it can be shown that $\{E(q) \mid q \in \mathbb{Q}\}$ is an "isomorphic copy" of $\mathbb{Q}$ in $\mathbb{R}$.

12 Decimal expansions

We now define the connection between real numbers and infinite decimal expansions.

$$(*) n.a_1a_2 \dots a_t \dots \qquad n \in \mathbb{Z}$$

which do not end in infinitely many nines. (This is to avoid "duplicate expansions" such as

$$0.5000 \dots = 0.49999 \dots .)$$

First, to each decimal expansion of the form $(*)$ we associate the Dedekind cut

$$\{q \in \mathbb{Q} \mid \text{There exists } t \geq 1 \text{ such that } q < n.a_1a_2 \dots a_t\}$$

Now let $x \in \mathbb{R}$ be any Dedekind cut. We describe how to compute the corresponding decimal expansion.

Case 1 Suppose that there exists $q = n.a_1 \dots a_t \in \mathbb{Q}$ such that $x = E(q)$. Then the decimal expansion of x is

$$n.a_1 \dots a_t 000 \dots$$

Case 2 We compute the decimal expansion of x inductively.

Step 0 Let $n \in \mathbb{Z}$ be the greatest integer such that $n \in x$.

Step 1 Let $0 \leq a_1 \leq 9$ be the greatest number such that $n.a_1 \in x$.

Step 2 Let $0 \leq a_2 \leq 9$ be the greatest number such that $n.a_1a_2 \in x$.

...

Step $t+1$ Suppose inductively that $n.a_1a_2 \dots a_t$ have been defined. Let $0 \leq a_{t+1} \leq 9$ be the greatest number such that $n.a_1a_2 \dots a_ta_{t+1} \in x$.

13 Construction of $\mathbb{C}$

Finally we want to expand $\mathbb{R}$ to a larger set of numbers $\mathbb{C}$ in which the equation $x^2+1=0$ has a solution.

[Basic idea: let i be a solution of $x^2+1=0$. Thus $i^2=-1$. Then each $z \in \mathbb{C}$ should have the form

$$z = x + iy$$

for some unique $x, y \in \mathbb{R}$.]

Definition 13.1. The set of complex numbers is defined to be

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}.$$

First we want to define an addition operation on $\mathbb{C}$. [Note that $(a+ib)+(c+id) = (a+c)+i(b+d)$.]

Definition 13.2. We define the binary operation $+_{\mathbb{C}}$ on $\mathbb{C}$ by

$$\langle a, b \rangle +_{\mathbb{C}} \langle c, d \rangle = \langle a+c, b+d \rangle.$$

Definition 13.3. $0_{\mathbb{C}} = \langle 0, 0 \rangle$.

Now we want to define a multiplication operation on $\mathbb{C}$. [Note that

$$\begin{aligned} (a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc).] \end{aligned}$$

Definition 13.4. We define the binary operation $\cdot_{\mathbb{C}}$ on $\mathbb{C}$ by

$$\langle a, b \rangle \cdot_{\mathbb{C}} \langle c, d \rangle = \langle ac - bd, ad + bc \rangle.$$

Definition 13.5. $1_{\mathbb{C}} = \langle 1, 0 \rangle$.

Theorem 13.6. $\langle 0, 1 \rangle \cdot_{\mathbb{C}} \langle 0, 1 \rangle = -1_{\mathbb{C}}$.

Proof.

$$\begin{aligned} \langle 0, 1 \rangle \cdot_{\mathbb{C}} \langle 0, 1 \rangle &= \langle -1, 0 \rangle \\ &= -\langle 1, 0 \rangle \\ &= -1_{\mathbb{C}} \end{aligned}$$

□

Definition 13.7. Define the function $E \colon \mathbb{R} \to \mathbb{C}$ by $E(r) = \langle r, 0 \rangle$. Then $\{E(r) \mid r \in \mathbb{R}\}$ is an "isomorphic copy" of $\mathbb{R}$ in $\mathbb{C}$.