# 14 Cardinalities

**Question 14.1.** When do two (possibly infinite) sets $A, B$ have the same size?

**Definition 14.2 (Cantor).** The set $A$ is *equinumerous* to the set $B$, written $A \approx B$, iff there exists a bijection $f\colon A \to B$.

**Example 14.3.** Let $\mathbb{E} = \{0, 2, 4, \dots\}$ be the set of even natural numbers. Then $\omega \approx \mathbb{E}$.

*Proof.* We can define a bijection $f\colon \omega \to \mathbb{E}$ by $f(n) = 2n$. □

**Important remark** It is often difficult to explicitly define a bijection $f\colon \omega \to A$. But another technique is usually easier. Suppose that $f\colon \omega \to A$ is a bijection. For each $n \in \omega$, let $a_n = f(n)$. Then

$$a_0, a_1, a_2, \dots, a_n, \dots$$

is a list which contains each element of $A$ exactly once. Conversely, if such a list exists, then we can define a bijection $f\colon \omega \to A$ by $f(n) = a_n$. Thus $\omega \approx A$ iff we can enumerate the elements of $A$ in such a list.

**Example 14.4.** $\omega \approx \mathbb{Z}$.

*Proof.* We can list the elements of $\mathbb{Z}$ as follows:

$$0, 1, -1, 2, -2, \dots, n, -n, \dots$$

□

**Example 14.5.** $\omega \approx \mathbb{Q}$

*Proof.* We proceed in two steps.

**Step 1** We will first show that $\omega \approx \mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$. Consider an array where the $ij^{\text{th}}$ entry is the ratio $j/i$. Work through the array by travelling along lines of slope 1, adding each rational encountered to the list if it has not already occured earlier. The resulting list shows that $\omega \approx \mathbb{Q}^+$.

**Step 2** We have just shown that there exists a bijection $f\colon \omega \to \mathbb{Q}^+$. Hence we can list the elements of $\mathbb{Q}$ as follows:

$$0, f(0), -f(0), f(1), -f(1), \dots, f(n), -f(n), \dots$$

Hence $\omega \approx \mathbb{Q}$. □

**Theorem 14.6 (Cantor, 1873).** $\omega \not\approx \mathbb{R}$.

*Proof.* For the sake of contradiction, assume that $\omega \approx \mathbb{R}$. Then we can list the elements of $\mathbb{R}$; say

$$r_0, r_1, \dots, r_n, \dots$$

For each $n \in \omega$, let

$$r_n = z_n . a_{0,n} a_{1,n} a_{2,n} a_{3,n} \dots a_{t,n} \dots$$

be the decimal expansion of $r_n$. Thus $z_n \in \mathbb{Z}$ and $0 \leq a_{t,n} \leq 9$ for each $t \geq 0$. Consider the array where the $ij^{\text{th}}$ entry is $a_{i,j}$ and look at the diagonal entries $a_{t,t}$.

Define $s = 0.b_0 b_1 b_2 \dots b_t \dots \in \mathbb{R}$ by

$$\begin{aligned} b_t &= 7 \quad \text{if } a_{t,t} \neq 7 \\ &= 5 \quad \text{if } a_{t,t} = 7. \end{aligned}$$

Then for each $n \in \omega$, $s \neq r_n$ since $s$ and $r_n$ differ on their $n^{\text{th}}$ decimal place. But this contradicts our assumption that every element of $\mathbb{R}$ occurs on the list. Hence $\omega \not\approx \mathbb{R}$. □

**Definition 14.7.** The set $A$ is *dominated* by the set $B$, written $A \preceq B$, iff there exists an injection $f\colon A \to B$.

**Definition 14.8.** $A \prec B$ iff $A \preceq B$ and $A \not\approx B$.

**Theorem 14.9.** $\omega \prec \mathbb{R}$

*Proof.* We can define an injection $f\colon \omega \to \mathbb{R}$ by $f(n) = n$. Thus $\omega \preceq \mathbb{R}$. Since $\omega \not\approx \mathbb{R}$, $\omega \prec \mathbb{R}$. □

**Theorem 14.10 (Cantor 1973).** *If $A$ is any set, then $A \prec \mathcal{P}(A)$.*

*Proof.* We can define an injection $f\colon A \to \mathcal{P}(A)$ by $f(a) = \{a\}$. Hence $A \preceq \mathcal{P}(A)$.

To see that $A \not\approx \mathcal{P}(A)$, we must prove that there does *not* exist a bijection $g\colon A \to \mathcal{P}(A)$. Let $g\colon A \to \mathcal{P}(A)$ be any function. Define $B \subseteq A$ by

$$a \in B \quad \text{iff} \quad a \notin g(a).$$

Then $B \in \mathcal{P}(A)$; and for each $a \in A$, $B \neq g(a)$ since $B$ and $g(a)$ differ on whether they contain the element $a$. Hence $g$ is not a surjection. □

**Corollary 14.11.** $\omega \prec \mathcal{P}(\omega) \prec \mathcal{P}(\mathcal{P}(\omega)) \prec \mathcal{P}(\mathcal{P}(\mathcal{P}(\omega))) \prec \dots$ □

We now develop some general theory.

**Theorem 14.12.** *For any sets $A, B, C$, we have:*

- $A \approx A$
- *If $A \approx B$, then $B \approx A$.*

- *If $A \approx B$ and $B \approx C$, then $A \approx C$.* □

**Theorem 14.13 (Schröder-Bernstein).** *If $A \preceq B$ and $B \preceq A$, then $A \approx B$.*

**Proof delayed**

**Theorem 14.14 (Zermelo's Theorem).** *For any sets $A$ and $B$, either $A \preceq B$ or $B \preceq A$.*

**Proof delayed**

In fact we shall need to introduce another axiom before we can prove Zermelo's Theorem.

**Definition 14.15.**

- The set $A$ is *finite* iff there exists $n \in \omega$ such that $A \approx n$.
- The set $A$ is *countably infinite* iff $A \approx \omega$.
- The set $A$ is *countable* iff $A$ is finite or countably infinite.
- The set $A$ is *uncountable* iff $A$ is not countable.

**Example 14.16.**

- $\mathbb{Q}$ is countable.
- $\mathbb{R}$ is uncountable.
- $\mathcal{P}(\omega)$ is uncountable.

Before proving the Schröder-Bernstein Theorem, we shall give a number of applications.

**Theorem 14.17.** $\omega \approx \mathbb{Q}$.

*Proof.* First we define an injection $f\colon \omega \to \mathbb{Q}$ by $f(n) = n$. Thus $\omega \prec \mathbb{Q}$.

Next we define an injection $g\colon \mathbb{Q} \to \omega$ as follows. Note that each $0 \neq q \in \mathbb{Q}$ can be expressed uniquely in the form $q = \epsilon \frac{a}{b}$, where

- $a, b \in \omega$ are relatively prime
- $\epsilon = \pm 1$.

Hence we can define an injection $g\colon \mathbb{Q} \to \omega$ be

$$\begin{aligned} g(\epsilon \frac{a}{b}) &= 2^{\epsilon+1} 3^a 5^b \\ g(0) &= 1 \end{aligned}$$

Thus $\mathbb{Q} \preceq \omega$. By Schröder-Bernstein, $\omega \approx \mathbb{Q}$. □

Next we shall prove that $\mathbb{R} \approx \mathcal{P}(\omega)$. We shall make use of the following result.

**Lemma 14.18.** $(0,1) \approx \mathbb{R}$.

*Proof.* Let $f\colon (0,1) \to \mathbb{R}$ be the function defined by

$$f(x) = \tan(\pi x - (\pi/2)).$$

By Calc I, $f$ is a bijection. □

**Exercise 14.19.** If $a < b$ are real numbers, then

- $(a,b) \approx (0,1)$
- $[a,b] \approx (0,1)$.

**Theorem 14.20.** $\mathbb{R} \approx \mathcal{P}(\omega)$.

*Proof.* Since $\mathbb{R} \approx (0,1)$, it is enough to show that $(0,1) \approx \mathcal{P}(\omega)$. We will make use of the fact that each $r \in (0,1)$ has a unique decimal expansion

$$(*) \quad r = 0, a_0 a_1 a_2 \dots a_n \dots$$

which does not end in an infinite sequence of nines.

**Step 1** First we prove that $(0,1) \preceq \mathcal{P}(\omega)$. Let

$$p_0 = 2, p_1 = 3, p_2 = 5, \dots, p_n, \dots$$

be the increasing enumeration of the primes. Then we can define a function $f\colon (0,1) \to \mathcal{P}(\omega)$ as follows: If

$$r = 0, a_0 a_1 a_2 \dots a_n \dots \in (0,1)$$

then

$$f(r) = \{p_0^{a_0+1}, p_1^{a_1+1}, \dots, p_n^{a_n+1}, \dots\}$$

Clearly $f$ is an injection and so $(0,1) \preceq \mathcal{P}(\omega)$.

**Step 2** Now we show that $\mathcal{P}(\omega) \preceq (0,1)$. We define a function $g\colon \mathcal{P}(\omega) \to (0,1)$ as follows. If $\emptyset \neq S \in \mathcal{P}(\omega)$, then

$$g(S) = 0.s_0 s_1 \dots s_n \dots$$

where $s_n = 1$ if $n \in S$ and $s_n = 0$ if $n \notin S$. We define $g(\emptyset) = 0.5$. Clearly $g$ is an injection and so $\mathcal{P}(\omega) \preceq (0,1)$.

By Schröder-Bernstein, $(0,1) \approx \mathcal{P}(\omega)$. □

**Definition 14.21.** $\mathrm{Fin}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$.

**Remark 14.22.** Clearly we have that $\mathbb{N} \preceq \mathrm{Fin}(\mathbb{N}) \preceq \mathcal{P}(\mathbb{N})$.

**Theorem 14.23.** $\mathbb{N} \approx \mathrm{Fin}(\mathbb{N})$.

*Proof.* Define the function $f\colon \mathbb{N} \to \mathrm{Fin}(\mathbb{N})$ by $f(n) = \{n\}$. Clearly $f$ is an injection and so $\mathbb{N} \preceq \mathrm{Fin}(\mathbb{N})$.

Next we define a function $g\colon \mathrm{Fin}(\mathbb{N}) \to \mathbb{N}$ as follows. Let $p_0, p_1, p_2, \dots, p_n, \dots$ be the increasing enumeration of the primes. Suppose that $\emptyset \neq S \in \mathrm{Fin}(\mathbb{N})$ and let $S = \{s_0, s_1, \dots, s_n\}$, where $s_0 < s_1 < \dots < s_n$. Then

$$g(S) = p_0^{s_0+1} p_1^{s_1+1} \dots p_n^{s_n+1}.$$

Finally we define $g(\emptyset) = 0$. Clearly $g$ is an injection and so $\mathrm{Fin}(\mathbb{N}) \preceq \mathbb{N}$.

By Schröder-Bernstein, $\mathbb{N} \approx \mathrm{Fin}(\mathbb{N})$. □

Recall that is $A, B$ are sets, then

$$B^A = \{f \mid f\colon A \to B\}.$$

**Theorem 14.24.** $\mathbb{N}^{\mathbb{N}} \approx \mathcal{P}(\omega)$.

*Proof.* First we define a function $f\colon \mathcal{P}(\omega) \to \mathbb{N}^{\mathbb{N}}$ as follows. For each $S \subseteq \mathbb{N}$, the *characteristic function* of $S$ is the function

$$\chi_S\colon \mathbb{N} \to \{0, 1\}$$

defined by $\chi_S(n) = 1$ if $n \in S$ and $\chi_S(n) = 0$ if $n \notin S$. If $S \in \mathcal{P}(\mathbb{N})$, then we define $f(S) = \chi_S$. Clearly $f$ is an injection and so $\mathcal{P}(\mathbb{N}) \preceq \mathbb{N}^{\mathbb{N}}$.

Next we define a function $g\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\omega)$ as follows. Let $p_0, p_1, \dots, p_n, \dots$ be the increasing enumeration of the primes. If $\phi \in \mathbb{N}^{\mathbb{N}}$, then we define

$$g(\phi) = \{p_0^{\phi(0)+1}, p_1^{\phi(1)+1}, \dots, p_n^{\phi(n)+1}, \dots\}.$$

Clearly $g$ is an injection and so $\mathbb{N}^{\mathbb{N}} \preceq \mathcal{P}(\mathbb{N})$.

By Schröder-Bernstein, $\mathbb{N}^{\mathbb{N}} \approx \mathcal{P}(\mathbb{N})$. □

**Heuristic Principle** Let $A$ be a "naturally occurring" set.

- If each $a \in A$ is determined by a finite amount of data, then $A \approx \mathbb{N}$.
- If each $a \in A$ is determined by infinitely many "independent" pieces of data, then $A \approx \mathcal{P}(\mathbb{N})$.

**Exercise 14.25.** Let $\mathrm{Inj}(\mathbb{N})$ be the set of injective functions $\phi\colon \mathbb{N} \to \mathbb{N}$. Prove that $\mathrm{Inj}(\mathbb{N}) \approx \mathcal{P}(\mathbb{N})$.

**Exercise 14.26.** Prove that $\mathbb{N}\times\mathbb{N} \approx \mathbb{N}$.

**Continuum Hypothesis (CH)** If $A \subset \mathbb{R}$ is *any* infinite subset, then either $A \approx \mathbb{N}$ or $A \approx \mathbb{R}$.

**Important Remark** It is known that the axioms of set theory can neither prove nor disprove CH.

**Exercise 14.27.** If $A \approx B$ and $C \approx D$, then $A\times C \approx B\times D$.

**Theorem 14.28.** $\mathbb{R}\times\mathbb{R} \approx \mathbb{R}$.

*Proof.* We have already proven that $\mathbb{R} \approx (0,1)$. By the exercise, $\mathbb{R}\times\mathbb{R} \approx (0,1)\times(0,1)$. Hence it it enough to prove that $(0,1)\times(0,1) \approx (0,1)$.

First we define a function $f\colon (0,1) \to (0,1)\times(0,1)$ by $f(r) = \langle r, 1/2\rangle$. Clearly $f$ is an injection and so $(0,1) \preceq (0,1)\times(0,1)$.

Next we define a function $g\colon (0,1)\times(0,1) \to (0,1)$ as follows. Suppose that $\langle r, s\rangle \in (0,1)\times(0,1)$ and let

$$r = 0.a_0a_1a_2\dots a_n\dots$$

$$s = 0.b_0b_1b_2\dots b_n\dots$$

be the decimal expansions. Then

$$g(r,s) = 0.a_0b_0a_1b_1\dots a_nb_n\dots$$

Clearly $g$ is an injection and so $(0,1)\times(0,1) \preceq (0,1)$.

By Schröder-Bernstein, $(0,1)\times(0,1) \preceq (0,1)$. □