15 Proof of Schröder-Bernstein

Next we turn to the proof of the Schröder-Bernstein Theorem.

Exercise 15.1. If $h\colon A \to B$ is an injection and $C \subset A$, then $h[A \smallsetminus C] = h[A] \smallsetminus h[C]$.

Theorem 15.2 (Schröder-Bernstein). *If $A \preceq B$ and $B \preceq A$, then $A \approx B$.*

Proof. Since $A \preceq B$ and $B \preceq A$, there exist injections $f\colon A \to B$ and $g\colon B \to A$. Let $C = g[B]$. Then clearly $B \approx C$ and so it is enough to show that $A \approx C$. Let $h = g \circ f$. Then $h\colon A \to A$ is an injection.

Define inductively

$$\begin{aligned} A_0 &= A \\ A_{n+1} &= h[A_n] \end{aligned}$$

and

$$\begin{aligned} C_0 &= C \\ C_{n+1} &= h[C_n]. \end{aligned}$$

Define a function $k\colon A \to C$ by

$$\begin{aligned} k(x) &= h(x) \quad \text{if } x \in A_n \smallsetminus C_n \text{ for some } n \in \omega \\ &= x \quad \text{otherwise.} \end{aligned}$$

Claim. $k\colon A \to C$ is an injection.

Proof. Suppose that $a \neq a' \in A$. There are three cases to consider.

Case 1. Supose that there exist $n, m \in \omega$ such that $a \in A_n \smallsetminus C_n$ and $a' \in A_m \smallsetminus C_m$. Then $k(a) = h(a) \neq h(a') = k(a')$, since h is an injection.

Case 2. Suppose that $a, a' \notin A_n \smallsetminus C_n$ for all $n \in \omega$. Then $k(a) = a \neq a' = k(a')$.

Case 3. Suppose that $a \in A_n \smallsetminus C_n$ for some $n \in \omega$ and $a' \notin A_m \smallsetminus C_m$ for all $m \in \omega$. Then $k(a) = h(a) \in h[A_n \smallsetminus C_n] = h[A_n] \smallsetminus h[C_n] = A_{n+1} \smallsetminus C_{n+1}$. Since $k(a') = a' \notin A_{n+1} \smallsetminus C_{n+1}$ we have that $k(a) \neq k(a')$. □

Claim. $k\colon A \to C$ is a surjection.

Proof. Let $c \in C$. There are two cases to consider.

Case 1. Suppose that $c \notin A_n \smallsetminus C_n$ for all $n \in \omega$. Then $k(c) = c$.

Case 2. Suppose that $c \in A_n \smallsetminus C_n$ for some $n \in \omega$. Since $c \notin A_0 \smallsetminus C_0$, we have that $n = m + 1$ for some $m \in \omega$. Thus

$$\begin{aligned} c \in A_{m+1} \smallsetminus C_{m+1} &= h[A_m] \smallsetminus h[C_m] \\ &= h[A_m \smallsetminus C_m]. \end{aligned}$$

Hence there exists $a \in A_m \smallsetminus C_m$ such that $k(a) = h(a) = c$. □

Thus $k \colon A \to C$ is a bijection and so $A \approx C$. This completes the proof of Schröder-Bernstein. □

16 Cardinal Numbers

Promise/Preliminary Definition. For every set A, we will define a corresponding *cardinal number* $\operatorname{card} A$ so that the following conditions are satisfied:

(a) For any sets A and B,

$$\operatorname{card} A = \operatorname{card} B \quad \text{iff} \quad A \approx B.$$

(b) If A is a finite set, then $\operatorname{card} A$ is the unique natural number n so that $A \approx n$.

Notation. $\operatorname{card} \omega = \aleph_0$.

Remark 16.1. The first "few" cardinal numbers are

$$0, 1, 2, \dots, n, \dots, \aleph_0, \aleph_1, \dots, \aleph_n, \dots, \aleph_\omega, \aleph_{\omega+1}, \dots$$

Hence CH is the statement that $\operatorname{card} \mathbb{R} = \aleph_1$.

Notation. If $\operatorname{card} A = \kappa$, then we say that A has cardinality κ.

Definition 16.2. If κ, λ are cardinals, then we define

$$\kappa + \lambda = \operatorname{card}(K \cup L)$$

where K, L are disjoint sets such that $\operatorname{card} K = \kappa$ and $\operatorname{card} L = \lambda$.

Of course, we must check that cardinal addition is well defined. This is accomplished by the following lemma.

Lemma 16.3. *Suppose that:*

(a) $K_1 \approx K_2$ *and* $L_1 \approx L_2$ *and*

(b) $K_1 \cap K_2 = \emptyset = L_1 \cap L_2$.

Then $K_1 \cup L_1 \approx K_2 \cup L_2$.

Proof. Since $K_1 \approx K_2$ and $L_1 \approx L_2$, there exist bijections $f\colon K_1 \to K_2$ and $g\colon L_1 \to L_2$. Hence we can define a bijection $h\colon K_1 \cup L_1 \to K_2 \cup L_2$ by

$$\begin{aligned} h(x) &= f(x) \quad \text{if } x \in K_1 \\ &= g(x) \quad \text{if } x \in L_1 \end{aligned}$$

Thus $K_1 \cup L_1 \approx K_2 \cup L_2$. □

Theorem 16.4. $\aleph_0 + \aleph_0 = \aleph_0$.

Proof. Let $\mathbb{E} = \{2n \mid n \in \omega\}$ and $\mathbb{O} = \{2n+1 \mid n \in \omega\}$. Then clearly $\mathbb{E} \cap \mathbb{O} = \emptyset$ and $\mathbb{E} \approx \omega \approx \mathbb{O}$. Thus

$$\begin{aligned} \aleph_0 + \aleph_0 &= \operatorname{card}(\mathbb{E} \cup \mathbb{O}) \\ &= \operatorname{card}\omega \\ &= \aleph_0. \quad \square \end{aligned}$$

Theorem 16.5. $\aleph_0 + \operatorname{card}\mathbb{R} = \operatorname{card}\mathbb{R}$.

Proof. Let $K = \omega$ and $L = (-2, -1)$. Then $K \cap L = \emptyset$ and $L \approx \mathbb{R}$. Hence

$$\aleph_0 + \operatorname{card}\mathbb{R} = \operatorname{card}(K \cup L).$$

Since $K \cup L \subset \mathbb{R}$, we have that $K \cup L \preceq \mathbb{R}$. Since $\mathbb{R} \approx L$, there exists an injection $f\colon \mathbb{R} \to K \cup L$ and so $\mathbb{R} \preceq K \cup L$. By Schröder-Bernstein, $K \cup L \approx \mathbb{R}$. Thus

$$\aleph_0 + \operatorname{card}\mathbb{R} = \operatorname{card}(K \cup L) = \operatorname{card}\mathbb{R}.$$

□

Exercise 16.6.

(a) $\aleph_0 + 5 = \aleph_0$.

(b) $\operatorname{card}\mathbb{R} + \operatorname{card}\mathbb{R} = \operatorname{card}\mathbb{R}$.

Remark 16.7. Eventually we shall prove that if κ, λ are cardinals and at least one is infinite, then

$$\kappa + \lambda = \max\{\kappa, \lambda\}.$$

Definition 16.8. If κ, λ are cardinals, then we define

$$\kappa \cdot \lambda = \operatorname{card}(K \times L)$$

where K, L are any sets such that $\operatorname{card} K = \kappa$ and $\operatorname{card} L = \lambda$.

An earlier exercise implies that cardinal multiplication is well-defined.

Theorem 16.9. $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Proof. By an earlier exercise, $\omega \times \omega \approx \omega$. Hence

$$\begin{aligned} \aleph_0 \cdot \aleph_0 &= \operatorname{card}(\omega \times \omega) \\ &= \operatorname{card}(\omega) \\ &= \aleph_0. \end{aligned}$$

□

Theorem 16.10. $\operatorname{card} \mathbb{R} \cdot \operatorname{card} \mathbb{R} = \operatorname{card} \mathbb{R}$.

Proof. As above, using the fact that $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$. □

Theorem 16.11. $\aleph_0 \cdot \operatorname{card} \mathbb{R} = \operatorname{card} \mathbb{R}$.

Proof. It is enough to show that $\omega \times \mathbb{R} \approx \mathbb{R}$. Since $\omega \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$, it follows that there exists an injection $f \colon \omega \times \mathbb{R} \to \mathbb{R}$ and so $\omega \times \mathbb{R} \preceq \mathbb{R}$. By By Schröder-Bernstein, $\omega \times \mathbb{R} \approx \mathbb{R}$. □

Remark 16.12. Eventually we shall prove that if κ, λ are nonzero cardinals and at least one is infinite, then

$$\kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

Definition 16.13. If κ, λ are cardinals, then we define

$$\kappa^\lambda = \operatorname{card}(K^L)$$

where K, L are any sets such that $\operatorname{card} K = \kappa$ and $\operatorname{card} L = \lambda$.

The following lemma shows that cardinal exponentiation is well-defined.

Lemma 16.14. *If $A \approx B$ and $C \approx D$, then $C^A \approx D^B$.*

Proof. Since $A \approx B$ and $C \approx D$, there exists bijections $f \colon A \to B$ and $g \colon C \to D$. Hence we can define a function

$$\pi \colon C^A \to D^B$$

by

$$\pi(\phi) = g \circ \phi \circ f^{-1}.$$

It is “easily checked” that π is a bijection. □

Theorem 16.15. *For any set A,* $\text{card}(\mathcal{P}(A)) = 2^{\text{card}\,A}$.

Proof. Note that $2^{\text{card}\,A} = \text{card}(\{0,1\}^A)$. Hence we must show that $\mathcal{P}(A) \approx \{0,1\}^A$. To see this, note that we can define a bijection

$$f\colon \mathcal{P}(A) \to \{0,1\}^A$$

by

$$f(S) = \chi_S$$

where $\chi_S\colon A \to \{0,1\}$ is the characteristic function defined by

$$\begin{aligned} \chi_S(a) &= 1 \quad \text{if } a \in S \\ &= 0 \quad \text{if } a \notin S \end{aligned}$$

□

Corollary 16.16. *If κ is any cardinal, then $2^\kappa \neq \kappa$.*

Proof. Let $\text{card}\,A = \kappa$. Then $\text{card}(\mathcal{P}(A)) = 2^\kappa$. By Cantor's Theorem, $A \not\approx \mathcal{P}(A)$ and so $\kappa \neq 2^\kappa$. □

Corollary 16.17. $\text{card}\,\mathbb{R} = 2^{\aleph_0}$

Proof. We have already proved that $\mathbb{R} \approx \mathcal{P}(\omega)$. Thus

$$\text{card}\,\mathbb{R} = \text{card}(\mathcal{P}(\omega)) = 2^{\text{card}\,\omega} = 2^{\aleph_0}.$$

□

Remark 16.18. Thus CH is the statement that $2^{\aleph_0} = \aleph_1$.

Exercise 16.19. $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.

Definition 16.20. If κ, λ are cardinals, then we define

$$\kappa \leq \lambda \quad \text{iff} \quad K \preceq L$$

where K, L are any sets such that $\text{card}\,K = \kappa$ and $\text{card}\,L = \lambda$.

The next lemma shows that this notion is well-defined.

Lemma 16.21. *Suppose that $A \approx B$ and $C \approx D$. Then $A \preceq C$ iff $B \preceq D$.*

Proof. Since $A \approx B$ and $C \approx D$, there exists bijections $f\colon A \to B$ and $g\colon C \to D$. Suppose that $A \preceq C$. Then there exists an injection $h\colon A \to C$. Hence we can define an injection $k\colon B \to D$ by $k = g \circ h \circ f^{-1}$. Thus $B \preceq D$. Similarly, if $B \preceq D$, then $A \preceq B$. □

Remark 16.22.

- The Schröder-Bernstein Theorem implies that if κ, λ are cardinals such that $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.
- Zermelo's Theorem implies that if κ, λ are cardinals, then either $\kappa \leq \lambda$ or $\lambda \leq \kappa$.

Definition 16.23. If κ, λ are cardinals, then we define

$$\kappa < \lambda \quad \text{iff} \quad \kappa \leq \lambda \quad \text{and} \quad \kappa \neq \lambda.$$

Remark 16.24. In other words, $\operatorname{card} K < \operatorname{card} L$ iff $K \prec L$.

Theorem 16.25. *For every cardinal κ, we have $\kappa < 2^\kappa$.*

Proof. Let $\operatorname{card} A = \kappa$. Then $A \prec \mathcal{P}(A)$. Hence

$$\kappa = \operatorname{card} A < \operatorname{card} \mathcal{P}(A) = 2^\kappa.$$

□