17 Axiom of Choice

Definition 17.1. Let $\mathcal{F}$ be a nonempty set of nonempty sets. Then a *choice function* for $\mathcal{F}$ is a function f such that $f(S) \in S$ for all $S \in \mathcal{F}$.

Example 17.2. Let $\mathcal{F} = \mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}$. Then we can define a choice function f by

$$f(S) = \text{the least element of } S.$$

Example 17.3. Let $\mathcal{F} = \mathcal{P}(\mathbb{Z}) \smallsetminus \{\emptyset\}$. Then we can define a choice function f by

$$f(S) = \epsilon n$$

where $n = \min\{|z| \mid z \in S\}$ and, if $n \neq 0$, $\epsilon = \min\{z/|z| \mid |z| = n, z \in S\}$.

Example 17.4. Let $\mathcal{F} = \mathcal{P}(\mathbb{Q}) \smallsetminus \{\emptyset\}$. Then we can define a choice function f as follows. Let $g \colon \mathbb{Q} \to \mathbb{N}$ be an injection. Then

$$f(S) = q$$

where $g(q) = \min\{g(r) \mid r \in S\}$.

Example 17.5. Let $\mathcal{F} = \mathcal{P}(\mathbb{R}) \smallsetminus \{\emptyset\}$. Then it is *impossible* to explicitly define a choice function for $\mathcal{F}$.

Axiom 17.6 (Axiom of Choice (AC)). *For every set $\mathcal{F}$ of nonempty sets, there exists a function f such that $f(S) \in S$ for all $S \in \mathcal{F}$.*

We say that f is a choice function for $\mathcal{F}$.

Theorem 17.7 (AC). *If A, B are non-empty sets, then the following are equivalent:*

(a) $A \preceq B$

(b) There exists a surjection $g \colon B \to A$.

Proof. $(a) \Rightarrow (b)$ Suppose that $A \preceq B$. Then there exists a injection $f \colon A \to B$. Fix some $a_0 \in A$. Then we can define a surjection $g \colon B \to A$ by

$$\begin{aligned} g(b) &= \text{the unique } a \in A \text{ such that } f(a) = b, \text{ if such an } a \text{ exists} \\ &= a_0, \quad \text{if no such } a \text{ exists.} \end{aligned}$$

$(b) \Rightarrow (a)$ Suppose that $g \colon B \to A$ is a surjection. Then for each $a \in A$,

$$S_a = \{b \in B \mid g(b) = a\} \neq \emptyset$$

Let $\mathcal{F} = \{S_a \mid a \in A\}$ and let h be a choice function for $\mathcal{F}$, *ie* $h(S_a) \in S_a$ for all $a \in A$. Then we can define a function $k \colon A \to B$ by $k(a) = h(S_a)$. We claim that k is an injection. To see this, suppose that $a \neq a' \in A$. Let $k(a) = b$ and $k(a') = b'$. Then $b \in S_a$, $b' \in S_{a'}$ and so $g(b) = a$, $g(b') = a'$. Hence $b \neq b'$. □

Theorem 17.8. *If A is any infinite set, then $\omega \preceq A$.*

Proof. Let f be a choice function for $\mathcal{F} = \mathcal{P}(A) \smallsetminus \{\emptyset\}$. Then we can define a function

$$g\colon \omega \to A$$

by recursion via

$$g(0) = f(A)$$

and

$$g(n+1) = f(A \smallsetminus \{g(0), \dots, g(n)\}).$$

Clearly g is an injection and so $\omega \preceq A$. □

Corollary 17.9 (AC). *If κ is any infinite cardinal, then $\aleph_0 \leq \kappa$.*

Proof. Let $\operatorname{card} A = \kappa$. Then A is an infinite set and so $\omega \preceq A$. Hence $\aleph_0 \leq \kappa$. □

Corollary 17.10 (AC). *A set A is infinite iff there exists a proper subset $B \subset A$ such that $B \approx A$.*

Proof. ($\Leftarrow$) If there exists a proper subset $B \subset A$ such that $B \approx A$, then A is clearly not finite. Hence A is infinite.
($\Rightarrow$) Suppose that A is infinite. Then $\omega \preceq A$ and so there exists an injection $f\colon \omega \to A$. Define a function $g\colon A \to A$ by

$$\begin{aligned} g(f(n)) &= f(n+1) \quad \text{for all } n \in \omega \\ g(x) &= x \quad \text{for all } x \notin \operatorname{ran} f. \end{aligned}$$

Then g is a bijection between A and $B = A \smallsetminus \{f(0)\}$. □

Corollary 17.11 (AC (Remark: doesn't really need AC)). *If A is a nonempty set, then A is countable iff there exists a surjection $f\colon \omega \to A$.*

Proof. This follows since A is countable iff $A \preceq \omega$. □

Theorem 17.12 (AC). *A countable union of countable sets is countable; ie if $\mathcal{A}$ is countable and each $A \in \mathcal{A}$ is countable, then $\bigcup \mathcal{A}$ is also countable.*

Proof. If $\mathcal{A} = \emptyset$, then $\bigcup \mathcal{A} = \emptyset$ is countable. Hence we can suppose that $\mathcal{A} \neq \emptyset$. We can also suppose that $\emptyset \notin \mathcal{A}$, since $\emptyset$ would contribute nothing to $\bigcup \mathcal{A}$.

Claim. There exists a surjection $f\colon \omega \times \omega \to \bigcup \mathcal{A}$.

Proof. Since $\mathcal{A}$ is countable, there exists a surjection $g\colon \omega \to \mathcal{A}$. For each $n \in \omega$, let $A_n = g(n)$. Then

$$\mathcal{A} = \{A_0, A_1, \dots, A_n, \dots\}.$$

(Of course, the sets A_n are not necessarily distinct!) Since each A_n is countable, we can choose a surjection $h_n\colon \omega \to A_n$. Then we can define a surjection

$$f\colon \omega\times\omega \to \bigcup \mathcal{A}$$

by

$$f(n,m) = h_n(m).$$

□

Finally, let $k\colon \omega \to \omega\times\omega$ be a surjection. Then $\varphi = f \circ k\colon \omega \bigcup \mathcal{A}$ is a surjection. Hence $\bigcup \mathcal{A}$ is countable. □

Question 17.13. Where did we use (AC) in the above proof?

Answer. Since each A_n is countable, we have that

$$S_n = \{h \mid h\colon \omega \to A_n \text{ is a surjection}\} \neq \emptyset.$$

We have applied (AC) to obtain a choice function for $\mathcal{F} = \{S_n \mid n \in \omega\}$.

Definition 17.14. If A is a set, then a *finite sequence* in A is a function $f\colon n \to A$, where $n \in \omega$.

Remark 17.15.

1. If $n = 0 = \emptyset$, then we obtain the *empty sequence*, $f = \emptyset$.
2. Suppose $n > 0$ and $f\colon n \to A$. For each $l \in n$, let $a_l = f(l)$. Then we often write f as
$$\langle a_0, a_1, \dots, a_{n-1}\rangle.$$

Definition 17.16. If A is a set, then $\mathrm{Sq}(A)$ is the set of finite sequences in A.

Theorem 17.17. $\mathrm{card}(\mathrm{Sq}(\omega)) = \aleph_0$.

Proof. We must show that $\mathrm{Sq}(\omega) \approx \omega$. First we can define an injection $g\colon \omega \to \mathrm{Sq}(\omega)$ by $g(n) = \langle n\rangle$. Thus $\omega \preceq \mathrm{Sq}(\omega)$. Next we can define an injection $h\colon \mathrm{Sq}(\omega) \to \omega$ as follows. Let $p_0, p_1, \dots, p_n, \dots$ be the increasing enumeration of the primes. Then

$$h(\emptyset) = 1$$

$$h(\langle a_0, \dots, a_{n-1}\rangle) = p_0^{a_0+1} p_1^{a_1+1} \dots p_{n-1}^{a_{n-1}+1}.$$

Thus $\mathrm{Sq}(\omega) \preceq \omega$. By Schröder-Bernstein, $\mathrm{Sq}(\omega) \approx \omega$. □

Corollary 17.18. *If A is a countable set, then* $\text{card}(\text{Sq}(A)) = \aleph_0$.

Proof. Once again, we must show that $\text{Sq}(A) \approx \omega$. Fix some $a \in A$. Then we define an injection $f\colon \omega \to \text{Sq}(A)$ by

$$\begin{aligned} f(n) &= \langle \overbrace{a, \ldots, a}^{n \text{ times}} \rangle, \quad n \geq 1 \\ &= \emptyset, \quad n = 0 \end{aligned}$$

Thus $\omega \preceq \text{Sq}(A)$. Next we can define an injection $h\colon \text{Sq}(A) \to \text{Sq}(\omega)$ as follows. Let $k\colon A \to \omega$ be an injection. Then

$$\begin{aligned} h(\emptyset) &= \emptyset \\ h(\langle a_0, a_1, \ldots, a_{n-1} \rangle) &= \langle k(a_0), k(a_1), \ldots, k(a_{n-1}) \rangle \end{aligned}$$

Thus $\text{Sq}(A) \preceq \text{Sq}(\omega)$. Since $\text{Sq}(\omega) \approx \omega$, it follows that $\text{Sq}(A) \approx \omega$. By Schröder-Bernstein, $\text{Sq}(A) \approx \omega$. □

18 Transcendental Numbers

Definition 18.1. Let $r \in \mathbb{R}$ be a real number.

(a) r is *algebraic* iff there exists a polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad a_n \neq 0$$

with integer coefficients such that $p(r) = 0$.

(b) Otherwise, r is *transcendental*.

Example 18.2.

(a) By considering $p(x) = 2x - 1$, we see that $1/2$ is algebraic. More generally, each $q \in \mathbb{Q}$ is algebraic.

(b) By considering $p(x) = x^2 - 2$, we see that $\sqrt{2}$ is algebraic.

(c) It is known (but hard to prove) that e and π are transcendental.

Theorem 18.3. *There exist uncountably many transcendental numbers.*

This is an easy corollary of the following result.

Theorem 18.4. *There exist only countably many algebraic numbers.*

Proof. Let P be the set of polynomials with integer coefficients. Then we can define an injection

$$f\colon P \to \mathrm{Sq}(\mathbb{Z})$$

by

$$f(a_0 + a_1 x + \ldots + a_n x^n) = \langle a_0, a_1, \ldots, a_n \rangle.$$

Thus $P \preceq \mathrm{Sq}(\mathbb{Z})$ and so P is countable. Note that each $p(x) \in P$ has finitely many roots. Thus the set of algebraic numbers is the union of countably many finite sets and hence is countable. □

Puzzle. Determine whether there exists:

(a) an uncountable set $\mathcal{A}$ of circular discs in $\mathbb{R}^2$, no two of which intersect.

(b) an uncountable set $\mathcal{B}$ of circles in $\mathbb{R}^2$, no two of which intersect.

(c) an uncountable set $\mathcal{C}$ of figure eights in $\mathbb{R}^2$, no two of which intersect.

19 Well-orderings

Definition 19.1. The set W is said to be *well-ordered* by $\prec$ iff:

(a) $\prec$ is a linear ordering of W; and

(b) every nonempty subset of W has a $\prec$–least element.

Example 19.2.

(a) The usual ordering $<$ on $\mathbb{N}$ is a well-ordering.

(b) The usual ordering $<$ on $\mathbb{Z}$ is *not* a well-ordering.

Theorem 19.3. *Let $\prec$ be a linear order on the set A. Then the following are equivalent:*

(a) $\prec$ is a well-ordering.

(b) There do not *exist elements $a_n \in A$ for $n \in \omega$ such that*

$$a_0 \succ a_1 \succ a_2 \succ \ldots \succ a_n \succ a_{n+1} \succ \ldots$$

Proof. $(a) \implies (b)$ We shall prove that not$(b) \implies$ not(a). So suppose that there exists elements $a_n \in A$ for $n \in \omega$ such that

$$a_0 \succ a_1 \succ \ldots \succ a_n \succ a_{n+1} \succ \ldots$$

Let $\emptyset \neq B = \{a_n \mid n \in \omega\} \subseteq A$. Then clearly B has *no* $\prec$–least element. Thus $\prec$ is *not* a well-ordering.

$(b) \implies (a)$ We shall prove that not$(a) \implies$ not(b). So suppose that $\prec$ is not a well-ordering. Then there exists $\emptyset \neq B \subseteq A$ such that B has *no* $\prec$–least element. Let $a_0 \in B$ be any element. Since a_0 is not the $\prec$–least element of B, there exists $a_1 \in B$ such that $a_1 \prec a_0$. Since a_1 is not the $\prec$–least element of B, there exists $a_2 \in B$ such that $a_2 \prec a_1$. Continuing in this fashion, we can recursively define elements $a_n \in B$ for $n \in \omega$ such that

$$a_0 \succ a_1 \succ a_2 \succ \ldots \succ a_n \succ a_{n+1} \succ \ldots$$

Thus (b) fails. □

Definition 19.4. Let $\langle L; < \rangle$ be a linearly ordered set.

(a) For each $a \in L$, the *set of predecessors* of a is defined to be

$$L[a] = \{b \in L \mid b < a\}.$$

(b) The subset S of L is an *initial segment* of L iff whenever $a \in S$ and $b < a$, then $b < a$.

(c) An initial segment S of L is *proper* iff $S \neq L$.

Remark 19.5. Thus S is an initial segment of L iff for all $a \in S$, we have $L[a] \subseteq S$.

Remark 19.6. If $\langle W; < \rangle$ is a well-ordering and $a \in W$, then so is $W[a]$.

Example 19.7. $(-\infty, 0]$ is an initial segment of $\mathbb{R}$. $(-\infty, 0)$ is an initial segment of $\mathbb{R}$.

Lemma 19.8. *If $\langle W; < \rangle$ is a well-ordering and S is a proper initial segment of W, then there exists $a \in W$ such that $S = W[a]$.*

Proof. Since S is proper, $W \smallsetminus S \neq \emptyset$. Let a be the $\prec$–least element of $W \smallsetminus S$. Then if $b \prec a$, then $b \in S$ and so $W[a] \subseteq S$. Suppose that there exists $c \in W \smallsetminus S$. Then $c \succeq a$. Since $c \in S$ and S is an initial segment of W, we must have that $a \in S$, which is a contradiction. Hence $W[a] = S$. □

Definition 19.9. Let $\langle L; < \rangle$ and $\langle M; \prec \rangle$ be (not necessarily distinct) linear orders.

(a) A function $f \colon L \to M$ is *order-preserving* iff for all $a, b \in L$

$$(*) \quad \text{if } a < b, \text{ then } f(a) \prec f(b).$$

(b) The function $f \colon L \to M$ is an *isomorphism* iff f is an order-preserving bijection. In this case, we say that L, M are *isomorphic* and write $L \cong M$.

Example 19.10. Define $f \colon \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x - 1$. Then f is an isomorphism.

Lemma 19.11. *If $\langle W; \prec \rangle$ is a well-ordering and $f \colon W \to W$ is order-presering, then $f(x) \succeq x$ for all $x \in W$.*

Proof. Suppose not. Then $C = \{x \in W \mid f(x) \preceq x\} \neq \emptyset$. Let a be the $\prec$–least element of C. Then $f(a) \prec a$ and so $f(f(a)) \prec f(a)$. But then $f(a) \in C$, which contradicts the fact that a is the $\prec$–least element of C. □

Remark 19.12. Define $f \colon (-\infty, \pi/2) \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= x, \quad \text{if } x < 0 \\ &= \tan(x), \quad \text{if } x \geq 0 \end{aligned}$$

Then f is an isomorphism. Hence $\mathbb{R}$ is isomorphic to its proper initial segment $(-\infty, \pi/2)$.

Lemma 19.13. *If $\langle W; \prec \rangle$ is a well-ordering, then W is* not *isomorphic to any of its proper initial segments.*

Proof. Suppose that S is a proper initial segment of W and that $f \colon W \to S$ is an isomorphism. By Lemma 19.8, there exists $a \in W$ such that $S = W[a]$. Since $f(a) \in W[a]$ we have that $f(a) \prec a$. Since f is order-preserving, this contradicts Lemma 19.11. □

Theorem 19.14. *Suppose that $\langle W_1; < \rangle$ and $\langle W_2; \prec \rangle$ are well-orderings. Then exactly one of the following holds:*

(a) W_1 and W_2 are isomorphic.

(b) W_1 is isomorphic to a proper initial segment of W_2.

(c) W_2 is isomorphic to a proper initial segment of W_1.

Before proving Theorem 19.14, we point out the connection with Zermelo's Theorem.

Definition 19.15. A set A is *well-orderable* iff there exists a well-ordering $\prec$ on A.

Example 19.16. Of course, $\langle \mathbb{Q}; < \rangle$ is *not* a well-ordering. However, $\mathbb{Q}$ is well-orderable.

Proof. Since $\mathbb{Q} \approx \omega$, there exists a bijection $f \colon \mathbb{Q} \to \omega$. Define a binary relation $\prec$ on $\mathbb{Q}$ by

$$a \prec b \quad \text{iff} \quad f(a) < f(b).$$

Then $\prec$ is a well-ordering of $\mathbb{Q}$. □

Problem 19.17. Is $\mathbb{R}$ well-orderable?

Corollary 19.18. *Suppose that A, B are well-orderable sets. Then either $A \preceq B$ or $B \preceq A$.*

Proof. Let $<$ be a well-ordering of A and let $\prec$ be a well-ordering of B. Applying Theorem 19.14 to $\langle A;<\rangle$ and $\langle B;\prec\rangle$, one of the following must hold.

(a) A and B are isomorphic

(b) A is isomorpic to a proper initial segment of B.

(c) B is isomorpic to a proper initial segment of A.

First suppose that (a) holds and let $f\colon A\to B$ be an isomorphism. Then f is a bijection and so $A\approx B$. Hence $A\preceq B$.

Now suppose (b) holds. Then there exists a proper initial segment $S\subset B$ and an isomorphism $f\colon A\to S$. Then f is an injection from A into B and so $A\preceq B$.

Similarly, if (c) holds, then $B\preceq A$. □

Later we shall use the Axiom of Choice to prove that every set is well-orderable.

Proof of Theorem 19.14. We break the proof down into a series of claims. Let $\langle W_1;<\rangle$ and $\langle W_2;<\rangle$ be well-orderings.

Claim 19.19. At most one of (a), (b), (c) holds.

Proof. Suppose, for example, that (a) and (b) hold. Thus there exists an isomorphism $f\colon W_1\to W_2$ and also an isomorphism $g\colon W_1\to S$, where S is a proper initial segment of W_2. But then $g\circ f^{-1}$ is an isomorphism, which contradicts Lemma 19.13. The other cases are similar. □

So it is enough to show that at least one of (a), (b), (c) holds. Define

$$f=\{\langle x,y\rangle\in W_1\times W_2 \mid W_1[x]\cong W_2[y]\}.$$

Claim 19.20. f is a function.

Proof. Suppose not. Then there exists $x\in W_1$ and $y\neq z\in W_2$ such that $\langle x,y\rangle,\langle x,z\rangle\in f$. Hence there exist isomorphisms $g\colon W_1[x]\to W_2[y]$ and $h\colon W_1[x]\to W_2[z]$. We can suppose that $y<z$, so that $W_2[y]$ is a proper initial segment of $W_2[z]$. But then $g\circ h^{-1}$ is an isomorphism, which contradicts Lemma 19.13. □

Claim 19.21.

(i) $\operatorname{dom} f$ is an initial segment of W_1.

(ii) f is order-preserving.

(iii) $\operatorname{ran} f$ is an initial segment of W_2.

Proof. Suppose that $a \in \operatorname{dom} f$ and $b < a$. Let $h\colon W_1[a] \to W_2[f(a)]$ be an isomorphism. Then $h|W_1[b]$ is an isomorphism betweem $W_1[b]$ and $W_2[h(b)]$. Thus $b \in \operatorname{dom} f$ and $f(b) = h(b) < f(a)$. Thus (i) and (ii) hold.

Finally suppose that $c \in \operatorname{ran} f$ and $d < c$. Then there exists $a \in W_1$ such that $\langle a, c\rangle \in f$. Let $h\colon W_1[a] \to W_2[c]$ be an isomorphism. Then there exists $b \in W_1[a]$ such that $h(b) = d$. Since $h|W_1[b]$ is an isomorphism between $W_1[b]$ and $W_2[d]$, it follows that $\langle b, d\rangle \in f$. Thus (iii) holds. □

Thus f is an order-preserving bijection between the initial segment $\operatorname{dom} f$ of W_1 and the initial segment $\operatorname{ran} f$ of W_2. There are four cases to consider.

Case 1. Suppose that $\operatorname{dom} f = W_1$ and $\operatorname{ran} f = W_2$. Then (a) holds.

Case 2. Suppose that $\operatorname{dom} f = W_1$ and $\operatorname{ran} f \neq W_2$. Then (b) holds.

Case 3. Suppose that $\operatorname{dom} f \neq W_1$ and $\operatorname{ran} f = W_2$. Then (c) holds.

Case 4. Suppose that $\operatorname{dom} f \neq W_1$ and $\operatorname{ran} f \neq W_2$. By Lemma 19.8, there exist $a \in W_1 \smallsetminus \operatorname{dom} f$ and $b \in W_2 \smallsetminus \operatorname{ran} f$ such that $\operatorname{dom} f = W_1[a]$ and $\operatorname{ran} f = W_2[b]$. But then $f\colon W_1[a] \to W_2[b]$ is an isomorphism, which means that $\langle a, b\rangle \in f$, a contradiction! □