20 Ordinals

Definition 20.1. A set α is an *ordinal* iff:

(i) α is transitive; and

(ii) α is linearly ordered by $\in$.

Example 20.2.

(a) Each natural number n is an ordinal.

(b) ω is an ordinal.

(a) $\omega \cup \{\omega\}$ is an ordinal.

At this point we require another axiom.

Axiom 20.3. *Every nonempty set A has an $\in$–minimal element; ie an element $x \in A$ such that $x \cap A = \emptyset$.*

We record some easy consequences of the Regularity Axiom.

Theorem 20.4.

(a) There does not exist an infinite sequence of sets such that

$$x_0 \ni x_1 \ni x_2 \ni \dots \ni x_n \ni x_{n+1} \ni \dots$$

(b) For every set x, we have $x \notin x$.

(c) There do not exist sets such that

$$x_0 \in x_1 \in x_2 \in \dots \in x_n \in x_0.$$

Proof. (a) Suppose that such an infinite sequence exits

$$x_0 \ni x_1 \ni x_2 \ni \dots \ni x_n \ni x_{n+1} \ni \dots$$

Let $A = \{x_n \mid n \in \omega\}$. For each $n \in \omega$, we have $x_{n+1} \in x_n \cap A \neq \emptyset$. But this means that A has no $\in$–minimal element, which is a contradiction.

(b) Suppose that $x \in x$. Then $A = \{x\}$ contradicts the Regularity Axiom.

(c) Exercise. □

Theorem 20.5. *If α is an ordinal, then α is well-ordered by $\in$.*

Proof. Consider the linear order $\langle \alpha; \in \rangle$. Since there do *not* exist elements $a_n \in \alpha$ for $n \in \omega$ such that

$$\alpha_0 \ni \alpha_1 \ni \alpha_2 \ni \ldots \ni \alpha_n \ni \alpha_{n+1} \ni \ldots$$

it follows that $\in$ is a well-ordering of α. □

Proposition 20.6. *Suppose that α is an ordinal.*

(a) $\alpha^+ = \alpha \cup \{\alpha\}$ *is also an ordinal.*

(b) If $\beta \in \alpha^+$, *then* $\beta \in \alpha$ *or* $\beta = \alpha$.

Proof. (b) is completely obvious! Thus it is enough to show that α^+ is an ordinal. Suppose that $\beta \in \alpha$ and $\gamma \in \beta$. Then either $\beta \in \alpha$ or $\beta = \alpha$. If $\beta \in \alpha$ then $\gamma \in \alpha$, since α is transitive, and so $\gamma \in \alpha^+ = \alpha \cup \{\alpha\}$. If $\beta = \alpha$, then $\gamma \in \alpha$ and so $\gamma \in \alpha^+ = \alpha \cup \{\alpha\}$. Hence α^+ is transitive.

Exercise 20.7. Show that α^+ is linearly ordered by $\in$.

Hence α^+ is also an ordinal. □

The ordinals begin as follows:

$$0, 1, 2, \ldots, n, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+n, \ldots, \omega \cdot 2,$$
$$\ldots, \omega \cdot 3, \ldots, \omega \cdot n, \ldots, \omega \cdot \omega, \ldots, \omega \cdot \omega \cdot \omega, \ldots$$
$$\ldots, \omega^n, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, etc$$

Question 20.8. Does there exist an uncountable ordinal?

Discussion. We could attempt to well-order $\mathbb{R}$ by "counting along the ordinals". But would we run out of ordinals before we finish?

Theorem 20.9. *If $\langle W; < \rangle$ is a well-ordering, then there exists a* unique *ordinal α such that $\langle W; < \rangle \cong \langle \alpha; \in \rangle$.*

First we need to prove a series of lemmas concerning the basic properties of ordinals. These are analogues of earlier results about the natural numbers.

Lemma 20.10. *If α is an ordinal and $\beta \in \alpha$, then β is also an ordinal and β is the set of predecessors of β in $\langle \alpha; \in \rangle$.*

Proof. Since $\beta \in \alpha$ and α is transitive, it follows that $\beta \subseteq \alpha$. Clearly $\in$ also linearly orders the subset β of α and hence β is an ordinal.

Exercise 20.11. Check that β is transitive!

Finally $\beta = \{\gamma \in \alpha \mid \gamma \in \beta\}$ is the set of predecessors of β in α. □

Lemma 20.12. *If α and β are ordinals and $\alpha \cong \beta$, then $\alpha = \beta$.*

Proof. Suppose that $f : \alpha \to \beta$ is an isomorphism. We claim that $f(\gamma) = \gamma$ for all $\gamma \in \alpha$. If not, let γ be the least element of α such that $f(\gamma) \neq \gamma$. Then

$$\begin{aligned} f(\gamma) &= \text{the set of predecessors of } f(\gamma) \text{ in } \beta \\ &= \{f(\xi) \mid \xi \in \gamma\}, \quad \text{since } f \text{ is an isomorphism} \\ &= \{\xi \mid \xi \in \gamma\}, \quad \text{by the minimality of } \gamma \\ &= \gamma \end{aligned}$$

which is a contradiction. □

Lemma 20.13. *If α and β are ordinals, then exactly one of the following holds:*

$$\alpha = \beta \quad \textit{or} \quad \alpha \in \beta \quad \textit{or} \quad \beta \in \alpha.$$

Proof. Since α, β are well-orders, exactly one of the following occurs:

(i) $\alpha \cong \beta$;

(ii) α is isomorphic to an initial segment of β;

(iii) β is isomorphic to an initial segment of α;

First suppose that (i) holds. By Lemma 20.12, we obtain that $\alpha = \beta$.

Next suppose that (ii) holds. Let S be a proper initial segment of β and let $f : \alpha \to S$ be an isomorphism. There exists $\gamma \in \beta$ such that

$$\begin{aligned} S &= \text{the set of predecessors of } \gamma \text{ in } \beta \\ &= \gamma \end{aligned}$$

Since $\alpha \cong \gamma$, Lemma 20.12 implies that $\alpha = \gamma \in \beta$.

Similarly, if (iii) holds, then $\beta \in \alpha$. □

Lemma 20.14. *If α, β, γ are ordinals and $\alpha \in \beta$ and $\beta \in \gamma$, then $\alpha \in \gamma$.*

Proof. This follows from the fact that γ is a transitive set. □

Definition 20.15. ON is the class of all ordinals.

Theorem 20.16. ON *is* not *a set.*

Proof. Suppose that ON is a set. By Lemma 20.10, ON is transitive. By Lemmas 20.14 and 20.13, ON is linearly ordered by $\in$. Thus ON is an ordinal and so ON $\in$ ON, which contradicts the Regularity Axiom. □

Arguing as above, we obtain the following result.

Lemma 20.17. *If A is a transitive set of ordinals, then A is an ordinal.* □

At this point, we require our final axiom.

Axiom 20.18 (Replacement). *Suppose that $P(x, y)$ is a property and A is a set. Suppose that for every $a \in A$, there exists a unique set b such that $P(a, b)$ holds. Then*

$$B = \{b \mid (\exists a \in A) \quad P(a, b)\}$$

is a set.

Theorem 20.19. *If $\langle W; < \rangle$ is a well-ordering, then there exists a unique ordinal α such that $W \cong \alpha$.*

Proof. By Lemma 20.12, there exists at most one such ordinal. Hence it is enough to prove the existance of at least one such ordinal. Define

$$A = \{a \in W \mid \text{There exists an ordinal } \beta \text{ such that } W[a] \cong \beta\}$$

By Lemma 20.12, for each $a \in A$, there exists a unique ordinal β such that $W[a] \cong \beta$. Since A is a set, the Replacement Axiom implies that

$$B = \{\beta \mid \text{There exists } a \in W \text{ such that } W[a] \cong \beta\}$$

is also a set. Let $f \colon A \to B$ be the function defined by

$$f(a) = \text{the unique ordinal } \beta \text{ such that } W[a] \cong \beta.$$

Claim 20.20.

(i) A is an initial segment of W.

(i) B is an ordinal.

(i) $f \colon A \to B$ is an isomorphism.

Proof. Suppose that $a \in A$ and that $b < a$. Let $f(a) = \alpha$ and let $h \colon W[h] \to \alpha$ be an isomorphism. Then $h|W[b]$ is an isomorphism between $W[b]$ and the set of predecessors of $h(b)$ in α; *ie* between $W[b]$ and $h(b) \in \alpha$. Thus $b \in A$ and $f(b) \in h(b) \in f(a)$.

A similar argument shows that if $\alpha \in B$ and $\beta \in \alpha$, then $\beta \in B$. Thus B is a transitive set of ordinals. By Lemma 20.17, B is an ordinal.

Finally, then first paragraph shows that f is order-preserving. Clearly f is a bijection and hence f is an isomorphism. □

Hence it is enough to prove that $A = W$. If not, A is a proper initial segment of W and so there exists $a \in W \smallsetminus A$ such that $A = W[a]$. But since $W[a] = A \cong B$ and B is an ordinal, this means that $a \in A$, which is a contradiction. □

Question 20.21. Does there exist an uncountable ordinal?

Answer. Yes! In fact, the following much stronger statement holds.

Theorem 20.22 (Hartogs). *For each set A, there exists an ordinal α such that $\alpha \not\preceq A$; ie such that there does* not *exist an injection $f \colon \alpha \to A$.*

Proof. Suppose that $\beta \in \mathrm{ON}$ and that $f \colon \beta \to A$ is an injection. Let $B = \operatorname{ran} f$. Then we can well-order B via

$$b_1 <_\beta b_2 \quad \text{iff} \quad f^{-1}(b_1) \in f^{-1}(b_2).$$

Hence $f \colon \beta \to \langle B; <_\beta \rangle$ is an isomorphism. Note

$$\langle B; <_\beta \rangle \in \mathcal{P}(A) \times \mathcal{P}(A \times A).$$

In fact, $\langle B; <_\beta \rangle$ is an element of the *set*

$$C = \{\langle S; < \rangle \in \mathcal{P}(A) \times \mathcal{P}(A \times A) \mid < \text{ is a well-ordering of } S\}.$$

Futhermore, for each $\langle S; < \rangle \in C$, there exists a unique ordinal γ such that $\langle S, < \rangle \cong \gamma$. By the Replacement Axiom,

$$D = \{\gamma \mid \text{There exists } \langle S, < \rangle \in C \text{ such that } \langle S, < \rangle \cong \gamma\}$$

is a set. Since ON is not a set, there exists an ordinal α such that $\alpha \notin D$. By the above argument, $\alpha \not\preceq A$. □

Remark 20.23. Note that an ordinal α is countable iff $\alpha \preceq \omega$. Hence there exists an uncountable ordinal. In fact, there exists a least such ordinal. Let ω_1 be the least uncountable ordinal.

Question 20.24. What exactly is ω_1?

Recall that the first finite ordinal is

$$\omega = \{\alpha \mid \alpha \text{ is a finite ordinal}\}.$$

Claim 20.25.

$$\omega_1 = \{\alpha \mid \alpha \text{ is a countable ordinal}\}.$$

Proof. By the minimality of ω_1, if $\alpha \in \omega_1$, then α is a countable ordinal. Conversely, suppose that α is a countable ordinal. Clearly $\alpha \neq \omega_1$. Hence either $\omega_1 \in \alpha$ or $\alpha \in \omega_1$. Suppose that $\omega_1 \in \alpha$. Since α is transitive, this implies that $\omega_1 \subseteq \alpha$, which is a contradiction, since ω_1 is uncountable and α is countable! □

Remark 20.26. Zermelo's Theorem $\Rightarrow$ (WA).

Proof. Let A be any set, Then there exists an ordinal α such that $\alpha \not\preceq A$. By Zermelo's Theorem, there exists an injection

$$f \colon A \to \alpha.$$

Hence we can well-order A by

$$a_1 < a_2 \quad \text{iff} \quad f(a_1) < f(a_2).$$

□

Summary.

$$(AC) \Leftarrow (WA) \Leftrightarrow (Z)$$

Theorem 20.27 (Transfinite induction on ON). *If C is a nonempty subclass of ON, then C contains an $\in$–least element.*

Proof. Suppose not. Then we can inductively construct a sequence of elements $\alpha_n \in C$ for $n \in \omega$ such that

$$\alpha_0 \ni \alpha_1 \ni \ldots \ni \alpha_n \ni \alpha_{n+1} \ni \ldots$$

But this contradicts the Axiom of Regularity. □

Notation. From now in, if α, β are ordinals, then we shall often write $\alpha < \beta$ instead of $\alpha \in \beta$.

Definition 20.28. Let $0 \neq \alpha \in$ ON.

- α is a *successor ordinal* iff there exists $\beta \in$ ON such that $\alpha = \beta^+$. We shall usually write $\beta + 1$ instead of β^+.
- Otherwise, α is a *limit ordinal.*

Example 20.29.

- $5, \omega^+$ are successor ordinals.
- ω is a limit ordinal.
- ω_1 is a limit ordinal.

Definition 20.30. If X is a set of ordinals, then $\sup(X)$ is the least ordinal α such that $\beta \leq \alpha$ for all $\beta \in X$.

eg,

$$\sup\{2,5,7\} = 7 = \cup\{2,5,7\}$$

$$\sup\{0,1,2,\dots,n,\dots\} = \omega = \cup\{0,1,2,\dots,n,\dots\}.$$

Exercise 20.31. If X is a set of ordinals, then $\sup(X) = \cup X$.

Proof. First we show that $\cup X$ is an ordinal. Clearly $\cup X$ is a set of ordinals. So it is enough to show that $\cup X$ is transitive. Suppose that $\alpha \in \beta$ and $\beta \in \cup X$. Then there exist $\gamma \in X$ such that $\beta \in \gamma$. Then $\alpha \in \gamma$ and so $\alpha \in \cup X$.

Next suppose that $\beta \in X$. Then $\beta \subseteq \cup X$ and so $\beta \leq \cup X$. Thus $\cup X$ is an upper bound for X.

Finally suppose that δ is an ordinal which is an upper bound for X. Then for all $\beta \in X$, we have $\beta \leq \delta$ and so $\beta \subseteq \delta$. Thus $\cup X \subseteq \delta$ and so $\cup X \leq \delta$. □

Theorem 20.32 (Transfinite Induction on ON). *Let D be a subclass of ON. Suppose that*

(a) $0 \in D$;

(b) If $\alpha \in D$, then $\alpha + 1 \in D$.

(c) If α is a limit ordinal and $\beta \in D$ for all $\beta < \alpha$,

Then $D = \mathrm{ON}$.

Proof. Suppose not. Then $C = \mathrm{ON} \setminus D$ is a nonempty class of ordinals. Hence C contains a least element α. But then $(a), (b), (c)$ imply that $\alpha \neq 0$, α isn't a successor ordinall and α isn't a limit ordinal, which is a contradiction! □

Next we shall consider transfinite recursion on ON.

Example 20.33. We define the operation of *ordinal addition* $\alpha + \beta$ by recursion on β as follows:

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \beta &= \sup\{\alpha + \gamma \mid \gamma < \beta\}, \quad \text{when } \beta \text{ is a limit ordinal.} \end{aligned}$$

Remark 20.34.

$$1 + \omega = \sup\{1 + n \mid n \in \omega\} = \omega.$$

Hence

$$\omega = 1 + \omega \neq \omega + 1 = \omega \cup \{\omega\}.$$

Thus ordinal addition is *not* commutative.

Example 20.35. We define the operation of *ordinal multiplication* $\alpha \cdot \beta$ by recursion on β as follows:

$$\begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \beta &= \sup\{\alpha \cdot \gamma \mid \gamma < \beta\}, \quad \text{when } \beta \text{ is a limit ordinal.} \end{aligned}$$

Remark 20.36. Thus

$$2 \cdot \omega = \sup\{2 \cdot n \mid n \in \omega\} = \omega.$$

Also

$$\begin{aligned} \omega \cdot 2 &= \omega \cdot (1 + 1) \\ &= \omega \cdot 1 + \omega \\ &= \omega + \omega \end{aligned}$$

Thus $2 \cdot \omega \neq \omega \cdot 2$. Hence ordinal multiplication is *not* commutative.

Theorem 20.37 (Transfinite Recursion). *If $G\colon V \to V$ is any operation, then there exists a unique operation $F\colon \mathrm{ON} \to V$ such that*

$$F(\alpha) = G(F|\alpha)$$

for all $\alpha \in \mathrm{ON}$

Proof. Omitted. □

Here V is the class of all sets.

Theorem 20.38 ((AC)). *Every set A is well-orderable.*

Proof. Let A be any set and let f be a choice function on $\mathcal{P}(A) \smallsetminus \{\emptyset\}$. Fix some $x \notin A$. Then we can define an operation

$$H\colon \mathrm{ON} \to A \cup \{x\}$$

by recursion as follows.

$$\begin{aligned} H(\alpha) &= f(A \smallsetminus \{H(\beta) \mid \beta < \alpha\}) \quad \text{if } A \smallsetminus \{H(\beta) \mid \beta < \alpha\} \neq \emptyset \\ &= x \quad \text{otherwise.} \end{aligned}$$

By Hartogs' Theorem, there exists an ordinal α such that $\alpha \not\preceq A$; and clearly $H(\alpha) = x$. Let β be the least ordinal such that $H(\beta) = x$ and let $h = H|\beta$. Then $h\colon \beta \to A$ is a bijection and so we can define a well-ordering $\prec$ of A by

$$a_1 \prec a_2 \quad \text{iff} \quad h^{-1}(a_1) < h^{-1}(a_2).$$

□

Finally we have shown

$$(WA) \Leftrightarrow (AC) \Leftrightarrow (Z)$$

At this point, we can pay off another of our debts!

Definition 20.39. For any set A, the *cardinal number* of A is

$$\operatorname{card} A = \text{the } \textit{least} \text{ ordinal } \alpha \text{ such that } A \approx \alpha.$$

Remark 20.40. Thus an ordinal γ is a cardinal number iff for all $\beta < \gamma$, $\beta \not\approx \gamma$.

Example 20.41. The following ordinals are cardinal numbers

$$0, 1, 2, \ldots, n, \ldots, \omega, \omega_1, \ldots$$

In particular, $\aleph_0 = \omega$.

Definition 20.42. We define $\aleph_\alpha$ by recursion as follows:

$$\begin{aligned}
\aleph_0 &= \omega \\
\aleph_{\alpha+1} &= \text{the least ordinal number } \kappa \text{ such that } \kappa \succ \aleph_\alpha. \\
\aleph_\delta &= \sup\{\aleph_\beta \mid \beta < \delta\}, \quad \text{if } \delta \text{ is a limit ordinal}
\end{aligned}$$

Then $\aleph_\alpha$, $\alpha \in \mathrm{ON}$, enumerates the infinite cardinal numbers in increasing order. **NB** This must be proved!

The Continuum Problem. We know that there exists an ordinal $\alpha \geq 1$ such that $2^{\aleph_0} = \aleph_\alpha$. But what is the exact value of α?

Theorem 20.43.

(a) Each $\aleph_\alpha$ *is an infinite cardinal.*

(b) If κ *is an infinite cardinal, then* $\kappa = \aleph_\alpha$ *for some* $\alpha \in \mathrm{ON}$.

Proof. (a) Suppose not. Let α be the least ordinal such that $\aleph_\alpha$ is not a cardinal. Clearly α is a limit ordinal and so

$$\aleph_\alpha = \sup\{\aleph_\beta \mid \beta < \alpha\}.$$

Hence there exists $\beta < \alpha$ such that $\aleph_\beta \approx \aleph_\alpha$. But then we have that

$$\aleph_{\beta+1} \subseteq \aleph_\alpha \approx \aleph_\beta$$

and so $\aleph_{\beta+1} \preceq \aleph_\beta$, which contradicts the fact that $\aleph_{\beta+1}$ is a cardinal number.

(b) Suppose not and let κ be the least counterexample. Clearly $\kappa > \aleph_0$. Let

$$S = \{\aleph_\beta \mid \aleph_\beta < \kappa\}.$$

Case 1. Suppose that S contains a maximum element; say $\aleph_\alpha$. Then

$$\begin{aligned} \kappa &= \text{the least cardinal greater than } \aleph_\gamma \\ &= \aleph_{\gamma+1} \end{aligned}$$

which is a contradiction!

Case 2. Suppose that S doesn't contain a maximum element. Then

$$\begin{aligned} \kappa &= \sup\{\aleph_\beta \mid \aleph_\beta \in S\} \\ &= \aleph_\alpha, \quad \text{where } \alpha = \sup\{\beta \mid \aleph_\beta \in S\} \end{aligned}$$

which is a contradiction! □

Definition 20.44. V is the class of all sets.

Theorem 20.45. *V is* not *a set.*

Proof. If V is a set, then $V \in V$, which contradicts the Regularity Axiom. □

Definition 20.46 (The cumulative hierarchy). We define V_α by recursion as follows:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\delta &= \bigcup\{V_\beta \mid \beta < \delta\}, \quad \text{if } \delta \text{ is a limit ordinal} \end{aligned}$$

Lemma 20.47.

(a) V_α is a transitive set for each $\alpha \in$ ON.

(b) If $\beta < \alpha$, then $V_\alpha \subseteq V_\beta$.

Proof. We argue by transfinite induction on α that (a) and (b) hold.

Case 1. $\alpha = 0$. Then (a) and (b) hold trivially.

Case 2. $\alpha = \gamma + 1$ is a successor ordinal. Suppose inductively that (a) and (b) hold for γ. Since V_γ is inductive, it follows that $V_{\gamma+1} = \mathcal{P}(V_\gamma)$ is transitive and that $V_\gamma \subseteq \mathcal{P}(V_\gamma) = V_{\gamma+1}$. Hence (a) holds. Now suppose that $\beta < \gamma + 1$. Then either $\beta = \gamma$ or $\beta < \gamma$. If $\beta = \gamma$, then we have that $V_\beta = V_\gamma \subseteq V_{\gamma+1}$. So suppose that $\beta < \gamma$. By induction hypothesis, $V_\beta \subseteq V_\gamma$. Since $V_\gamma \subseteq V_{\gamma+1}$, we have that $V_\beta \subseteq V_{\gamma+1}$. Thus (b) holds.

Case 3. α is a limit ordinal. Suppose inductively that (a) and (b) hold for all $\gamma < \alpha$. Since

$$V_\alpha = \bigcup_{\gamma<\alpha} V_\gamma$$

it follows that (b) holds. Next suppose that $x \in y$ and $y \in V_\alpha$. Then there exists $\gamma < \alpha$ such that $y \in V_\gamma$. Since V_γ is transitive, $x \in V_\gamma$. Since $V_\gamma \subseteq V_\alpha$, $x \in V_\alpha$. Thus (a) holds. □

Theorem 20.48. *If x is any set, then there exists $\alpha \in \mathrm{ON}$ such that $x \in V_\alpha$; ie*

$$V = \bigcup_{\alpha\in\mathrm{ON}} V_\alpha.$$

Proof. Suppose that $x \in V$ and that for each $y \in x$, there exists $\alpha_y \in \mathrm{ON}$ such that $y \in V_{\alpha_y}$. Let

$$\beta = \sup\{\alpha_y \mid y \in x\}.$$

Then $x \subseteq V_\beta$ and so $x \in \mathcal{P}(V_\beta) = V_{\beta+1}$.

For the sake of contradiction, suppose that there exists $x_0 \in V$ such that $x_0 \notin \bigcup_{\alpha\in\mathrm{ON}} V_\alpha$. Then there exists $x_1 \in x_0$ such that $x_1 \notin \bigcup_{\alpha\in\mathrm{ON}} V_\alpha$. Similarly, there exists $x_2 \in x_1$ such that $x_2 \notin \bigcup_{\alpha\in\mathrm{ON}} V_\alpha$. Continuing in this fashion, we can inductively define sets x_n such that

$$x_0 \ni x_1 \ni x_2 \ni \ldots \ni x_n \ni x_{n+1} \ni \ldots$$

which contradicts the Regularity Axiom. □

Remark 20.49.

(a) A class $M \subseteq V$ is a set iff there exists $\alpha \in \mathrm{ON}$ such that $M \subseteq V_\alpha$.

(b) For each ordinal $\alpha \in \mathrm{ON}$, we have that $\alpha \in V_{\alpha+1} \setminus V_\alpha$.