These notes are transcribed from class notes written by Professor Simon Thomas. The notes follow the notation of Enderton's *A mathematical introduction to logic* but may be read independently. Please contact Saul Schleimer if you find any errors.

1 A few introductory remarks

In mathematical reasoning, logical arguments are used to deduce the consequences (called *theorems*) of basic assumptions (called *axioms*).

Question 1.1. What does it mean for one sentence to "follow logically" from another sentence?

Question 1.2. Suppose that a sentence σ does *not* follow logically from the set T of axioms. How can we prove that this is so?

We will begin the course by studying some basic set theory.

Motivation:

1. We will need this material in our study of mathematical logic.
2. Set theory is a foundation for all of mathematics.
3. Set theory is beautiful.

Remark 1.3. In a couple of weeks we will come across a natural set-theoretic statement, the Continuum Hypothesis, which can neither be proved nor disproved using the classical axioms of set theory.

2 Basic Set Theory

Notation: $\{2, 3, 5\} = \{2, 5, 5, 2, 3\}$.
$\{0, 2, 4, 6, \ldots\} = \{x \mid x \text{ is an even natural number}\}$.
$x \in A$ means "x is an element of A".
$\emptyset$ is the empty set.
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z},\ b \neq 0\}$ is the set of rational numbers.
$\mathbb{R}$ is the set of real numbers.

Axiom of Extensionality: Suppose that A, B are sets. If for all x,

$$x \in A \text{ iff } x \in B$$

then $A = B$.

Definition 2.1. Suppose that A, B are sets. Then A is a *subset* of B, written $A \subseteq B$, iff for all x,

if $x \in A$ then $x \in B$.

Example 2.2.

1. $\mathbb{N} \subseteq \mathbb{Z}$
2. If A is any set then $\emptyset \subseteq A$.

Proposition 2.3. *If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

Proof. Let x be arbitrary. Since $A \subseteq B$, if $x \in A$ then $x \in B$. Since $B \subseteq A$, if $x \in B$ then $x \in A$. Hence $x \in A$ iff $x \in B$. By the Axion of Extensionality, $A = B$. □

Definition 2.4. Let A, B be sets. The *union* of A and B, written $A \cup B$, is the set defined by

$x \in A \cup B$ iff $x \in A$ or $x \in B$.

Proposition 2.5. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Let x be arbitrary. Then $x \in A \cup (B \cup C)$
iff $x \in A$ or $x \in B \cup C$
iff $x \in A$ or ($x \in B$ or $x \in C$)
iff $x \in A$ or $x \in B$ or $x \in C$
iff ($x \in A$ or $x \in B$) or $x \in C$
iff $x \in A \cup B$ or $x \in C$
iff $x \in (A \cup B) \cup C$. □

Definition 2.6. Let A, B be sets. The *intersection* of A and B, written $A \cap B$ is the set defined by

$x \in A \cup B$ iff $x \in A$ and $x \in B$.

Exercise 2.7. Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Proposition 2.8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. Let x be arbitrary. Then $x \in A \cap (B \cup C)$
iff $x \in A$ and $x \in B \cup C$
iff $x \in A$ and ($x \in B$ or $x \in C$)
iff ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)
iff ($x \in A \cap B$) or ($x \in A \cap C$)
iff $x \in (A \cap B) \cup (A \cap C)$. □

Exercise 2.9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Definition 2.10. Let A, B be sets. The *set theoretic difference* of A and B, written $A \smallsetminus B$, is the set defined by

$x \in A \smallsetminus B$ iff $x \in A$ and $x \notin B$.

$\{1,2,3\} \setminus \{3,4,5\} = \{1,2\}$.
$\mathbb{N} \setminus \mathbb{Z} = \emptyset$.
$\mathbb{Z} \setminus \mathbb{N} = \{-1,-2,-3,\ldots\}$.

Proposition 2.11. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Proof. Let x be arbitrary. Then $x \in A \setminus (B \cup C)$
iff $x \in A$ and $x \notin B \cup C$
iff $x \in A$ and not ($x \in B$ or $x \in C$)
iff $x \in A$ and ($x \notin B$ and $x \notin C$)
iff $x \in A$ and $x \notin B$ and $x \notin C$
iff $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$
iff ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)
iff $x \in A \setminus B$ and $x \in A \setminus C$
iff $x \in (A \setminus B) \cap (A \setminus C)$. □

Exercise 2.12. Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.