3 Functions

Provisional Definition:

Let A, B be sets. Then f is a *function* from A to B, written $f\colon A \to B$, iff f assigns a unique element $f(a) \in B$ to each $a \in A$.

What is the meaning of "assigns"? To illustrate our earlier comments on set theory as a foundation for mathematics, we shall reduce the notion of a function to the language of basic set theory.

Basic idea

For example, consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then the graph of f is a subset of $\mathbb{R}^2$. We shall identify f with its graph.

To generalize this idea to arbitrary functions, we first need to introduce the idea of an *ordered pair*; ie a mathematical object $\langle a, b\rangle$ such that

$(*)$ $\qquad \langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$.

Definition 3.1. Let A and B be sets. Then the *Cartesian product* of A and B is the set

$$A\times B = \{\langle a, b\rangle \mid a \in A,\ b \in B\}.$$

Definition 3.2. f is a function from A to B iff the following conditions hold:

1. $f \subseteq A\times B$
2. For each $a \in A$, there is a unique $b \in B$ such that $\langle a, b\rangle \in f$.

In this case, the unique such b is said to be the value of f at a and we write $f(a) = b$.

In order to reduce the notion of a function to basic set theory, we now only need to find a purely set theoretic object to play the role of $\langle x, y\rangle$.

Definition 3.3. $\qquad \langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

Finally, we must prove that with this definition, the set $\langle x, y\rangle$ satisfies $(*)$.

Theorem 3.4. *$\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$.*

Proof. ($\Leftarrow$): Clearly if $a = c$ and $b = d$ then $\langle a, b\rangle = \langle c, d\rangle$.

($\Rightarrow$): Conversely, suppose that $\langle a, b\rangle = \langle c, d\rangle$; ie

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

We split our analysis into three cases.

Case 1

Suppose that $a = b$. Then $\{\{a\}, \{a, b\}\}$ equals

$$= \{\{a\}, \{a, a\}\}$$
$$= \{\{a\}, \{a\}\}$$
$$= \{\{a\}\}$$

Since

$$\{\{c\}, \{c, d\}\} = \{\{a\}\}$$

it follows that

$$\{c\} = \{c, d\} = \{a\}.$$

This implies that $c = d = a$. Hence $a = c$ and $b = d$.

Case 2

Similarly, if $c = d$, we obtain that $a = c$ and $b = d$.

Case 3

Finally suppose that $a \neq b$ and $c \neq d$. Since

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

we must have that $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. Since $c \neq d$ the second option is impossible. Hence $\{a\} = \{c\}$ and so $a = c$.

Also $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. Clearly the first option is impossible and so $\{a, b\} = \{c, d\}$. Since $a = c$, we must have $b = d$. □

Important remark When working with functions, it is almost never necessary to remember that a function is literally a set of ordered pair as above.

Definition 3.5. The function $f\colon A \to B$ is an *injection* (one-to-one) iff

$$a \neq a' \text{ implies } f(a) \neq f(a').$$

Definition 3.6. The function $f\colon A \to B$ is a *surjection* (onto) iff for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition 3.7. If $f\colon A \to B$ and $g\colon B \to C$ are functions, then their *composition* is the function $g \circ f\colon A \to C$ defined by $(g \circ f)(a) = g(f(a))$.

Proposition 3.8. *If $f\colon A \to B$ and $g\colon B \to C$ are surjections then $g \circ f\colon A \to C$ is also a surjection.*

Proof. Let $c \in C$ be arbitrary. Since g is surjective, there exists a $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) =$

$$= g(f(a))$$
$$= g(b)$$
$$= c$$

Thus $g \circ f$ is surjective. □

Exercise 3.9. If $f\colon A \to B$ and $g\colon B \to C$ are injections then $g \circ f\colon A \to C$ is also an injection.

Definition 3.10. The function $f\colon A \to B$ is a *bijection* iff f is both an injection and a surjection.

Definition 3.11. If $f\colon A \to B$ is a bijection, then the *inverse* $f^{-1}\colon B \to A$ is the function defined by

$$f^{-1}(b) = \text{the unique } a \in A \text{ so that } f(a) = b.$$

Remark 3.12. 1. It is easily checked that $f^{-1}\colon B \to A$ is also a bijection.

2. In terms of ordered pairs:

$$f^{-1} = \{\langle b, a\rangle \;\mid\; \langle a, b\rangle \in f\}.$$

4 Equinumerosity

Definition 4.1. Two sets A and B are *equinumerous*, written $A \sim B$, iff there exists a bijection $f\colon A \to B$.

Example 4.2. Let $\mathbb{E} = \{0, 2, 4, \ldots\}$ be the set of even natural numbers. Then $\mathbb{N} \sim \mathbb{E}$.

Proof. We can define a bijection $f\colon \mathbb{N} \to \mathbb{E}$ by $f(n) = 2n$. □

Important remark It is often extremely hard to explicitly define a bijection $f\colon \mathbb{N} \to A$. But suppose such a bijection exists. Then letting $a_n = f(n)$, we obtain a *list* of the elements of A

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

in which each element of A appears exactly once. Conversely, if such a list exists, then we can define a bijection $f\colon \mathbb{N} \to A$ by $f(n) = a_n$.

Example 4.3. $\mathbb{N} \sim \mathbb{Z}$

Proof. We can list the elements of $\mathbb{Z}$ by

$$0, 1, -1, 2, -1, \ldots, n, -n, \ldots$$

□

Theorem 4.4. $\mathbb{N} \sim \mathbb{Q}$

Proof. **Step 1** First we prove that $\mathbb{N} \sim \mathbb{Q}^+$, the set of positive rational numbers. Form an infinite matrix where the $(i, j)^{\text{th}}$ entry is j/i.

Proceed through the matrix by traversing, alternating between upward and downward, along lines of slope one. At the $(i, j)^{\text{th}}$ entry add the number j/i to the list if it has not already appeared.

Step 2 We have shown that there exists a bijection $f\colon \mathbb{N} \to \mathbb{Q}^+$. Hence we can list the elements of $\mathbb{Q}$ by

$$0, f(0), -f(0), f(1), -f(1), \ldots$$

□

Definition 4.5. If A is any set, then its *powerset* is defined to be

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$

Example 4.6. 1. $\mathcal{P}(\{1,2\} = \{\emptyset, \{1\}, \{2\}, \{1,2\}$.

2. $\mathcal{P}(\{1, 2, \ldots n, \}$ has size 2^n.

Theorem 4.7. *(Cantor)* $\mathbb{N} \nsim \mathcal{P}(\mathbb{N})$

Proof. (The diagonal argument) We must show that there does *not* exist a bijection $f \colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$. So let $f \colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any function. We shall show that f isn't a surjection. To accomplish this we shall define a subset $S \subseteq \mathbb{N}$ such that $f(n) \neq S$ for all $n \in \mathbb{N}$. We do this via a "time and motion study". For each $n \in \mathbb{N}$, we must perform:

1. the n^{th} decision: is $n \in S$?

2. the n^{th} task: we must ensure that $f(n) \neq S$.

We decide to accomplish the n^{th} task with the n^{th} decision. So we decide that

$$n \in S \text{ iff } n \notin f(n).$$

Clearly S and $f(n)$ differ on whether they contain n and so $f(n) \neq S$. Hence f is not a surjection. □

Discussion Why is this called the "diagonal argument"?

Definition 4.8. A set A is *countable* iff A is finite or $\mathbb{N} \sim A$. Otherwise A is *uncountable*.

eg $\mathbb{Q}$ is countable
$\mathcal{P}(\mathbb{N})$ is uncounable.

Theorem 4.9. *(Cantor) If A is any set, then $A \nsim \mathcal{P}(A)$.*

Proof. Supose that $f \colon A \to \mathcal{P}(A)$ is any function. We shall show that f isn't a surjection. Define $S \subseteq A$ by

$$a \in S \text{ iff } a \notin f(a).$$

Then S and $f(a)$ differ on whether they contain a. Thus $f(a) \neq S$ for all $a \in A$. □

Definition 4.10. Let A, B be sets.

1. $A \preceq B$ iff there exists an injection $f \colon A \to B$.

2. $A \prec B$ iff $A \preceq B$ and $A \nsim B$.

Corollary 4.11. *If A is any set, then $A \prec \mathcal{P}(A)$.*

Proof. Define $f \colon A \to \mathcal{P}(A)$ by $f(a) = \{a\}$. Clearly f is an injection and so $A \preceq \mathcal{P}(A)$. Since $A \nsim \mathcal{P}(A)$, we have $A \prec \mathcal{P}(A)$. □

Corollary 4.12. $\mathbb{N} \prec \mathcal{P}(\mathbb{N}) \prec \mathcal{P}(\mathcal{P}(\mathbb{N})) \prec \ldots$ □

Having seen that we have a nontrivial subject, we now try to develop some general theory.

5 Cantor-Bernstein Theorem

Theorem 5.1. *Let A, B, C be sets.*

1. $A \sim A$
2. *If $A \sim B$, then $B \sim A$.*
3. *If $A \sim B$ and $B \sim C$, then $A \sim C$.*

□

Exercise 5.2. If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.

Theorem 5.3. *(Cantor-Bernstein) If $A \preceq B$ and $B \preceq A$, then $A \sim B$.*

Proof delayed

Theorem 5.4. *If A, B are any sets, then either $A \preceq B$ or $B \preceq A$.*

Proof omitted

This theorem is equivalent to:

Axiom of Choice If $\mathcal{F}$ is a family of nonempty sets then there exists a function f such that $f(A) \in A$ for all $A \in \mathcal{F}$. (Such a function is called a *choice function.*)