5 The Cantor-Bernstein Theorem (continued)

Some applications of the Cantor-Bernstein theorem

Theorem 5.1. $\mathbb{N} \sim \mathbb{Q}$.

Proof. First define a function $f\colon \mathbb{N} \to \mathbb{Q}$ by $f(n) = n$. Clearly f is an injection and so $\mathbb{N} \preceq \mathbb{Q}$.

Now define a function $g\colon \mathbb{Q} \to \mathbb{N}$ as follows. First suppose that $0 \neq q \in \mathbb{Q}$. Then we can *uniquely* express

$$q = \epsilon \tfrac{a}{b}$$

where $\epsilon = \pm 1$ and $a, b \in \mathbb{N}$ are positive and relatively prime. Then we define

$$g(q) = 2^{\epsilon+1} 3^a 5^b.$$

Finally define $g(0) = 7$. Clearly g is an injection and so $\mathbb{Q} \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \mathbb{Q}$. □

Theorem 5.2. $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$.

We shall make use of the following result.

Lemma 5.3. $(0, 1) \sim \mathbb{R}$.

Proof of Lemma 5.3. By Calc I, we can define a bijection $f\colon (0, 1) \to \mathbb{R}$ by $f(x) = \tan(\pi x - \pi/2)$. □

Proof of Theorem 5.2. By the lemma, it is enough to show that $(0, 1) \sim \mathcal{P}(\mathbb{N})$. We shall make use of the fact that eact $r \in (0, 1)$ has a *unique* decimal expansion

$$r = 0.r_1 r_2 r_3 \dots r_n \dots$$

so that

1. $0 \leq r_n \leq 9$
2. the expansion does not terminate with infinitely many 9s. (This is to avoid two expansions such as $0.5000\dots = 0.4999\dots$)

First we define $f\colon (0, 1) \to \mathcal{P}(\mathbb{N})$ as follows. If

$$r = 0.r_0 r_1 r_2 \dots r_n \dots$$

then

$$f(r) = \{2^{r_0+1}, 3^{r_1+1}, \dots, p_n^{r_n+1}, \dots\}$$

where p_n is the n^{th} prime. Clearly f is an injection and so $(0, 1) \preceq \mathcal{P}(\mathbb{N})$.

Next we define a function $g\colon \mathcal{P}(\mathbb{N}) \to (0, 1)$ as follows: If $\emptyset \neq S \subseteq \mathbb{N}$ then

$$g(S) = 0.s_0 s_1 s_2 \dots s_n \dots$$

where

$$s_n = 0 \text{ if } n \in S$$
$$s_n = 1 \text{ if } n \notin S.$$

Finally, $g(\emptyset) = 0.5$. Clearly g is an injection and so $\mathcal{P}(\mathbb{N}) \preceq (0, 1)$.

By Cantor-Bernstein, $(0, 1) \sim \mathcal{P}(\mathbb{N})$. □

The following result says that "$\mathbb{N}$ has the smallest infinite size."

Theorem 5.4. *If $S \subseteq \mathbb{N}$, then either S is finite or $\mathbb{N} \sim S$.*

Proof. Suppose that S is infinite. Let

$$s_0, s_1, s_2, \ldots, s_n, \ldots$$

be the increasing enumeration of the elements of S. This list witnesses that $\mathbb{N} \sim S$. □

The Continuum Hypothesis (CH) If $S \subseteq \mathbb{R}$, then either S is countable or $\mathbb{R} \sim S$.

Theorem 5.5. *(Godel 1930s, Cohen 1960s) If the axioms of set theory are consistent, then CH can neither be proved nor disproved from these axioms.*

Definition 5.6. Fin($\mathbb{N}$) is the set of all finite subsets of $\mathbb{N}$.

Theorem 5.7. $\mathbb{N} \sim \text{Fin}(\mathbb{N})$.

Proof. First define $f\colon \mathbb{N} \to \text{Fin}(\mathbb{N})$ by $f(n) = \{n\}$. Clearly f is an injection and so $\mathbb{N} \preceq \text{Fin}(\mathbb{N})$. Now define $g\colon \text{Fin}(\mathbb{N}) \to \mathbb{N}$ as follows. If $s = \{s_0, s_1, s_2, \ldots, s_n\}$ where $s_0 < s_1 < \ldots < s_n$, then

$$g(S) = 2^{s_0+1}3^{s_1+1} \ldots p_n^{s_n+1}$$

where p_i is the i^{th} prime. Also we define $g(\emptyset) = 1$. Clearly g is an injection and so $\text{Fin}(\mathbb{N}) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \text{Fin}(\mathbb{N})$. □

Exercise 5.8. If $a < b$ are reals, then $(a, b) \sim (0, 1)$.

Exercise 5.9. If $a < b$ are reals, then $[a, b] \sim (0, 1)$.

Exercise 5.10. $\mathbb{N} \sim \mathbb{N}\times\mathbb{N}$.

Exercise 5.11. If $A \sim B$ and $C \sim D$, then $A\times C \sim B\times D$.

Definition 5.12. If A and B are sets, then

$$B^A = \{f \mid f\colon A \to B\}.$$

Theorem 5.13. $\mathcal{P}(\mathbb{N}) \sim \mathbb{N}^{\mathbb{N}}$.

Proof. For each $S \subseteq \mathbb{N}$ we define the corresponding characteristic function $\chi_S\colon \mathbb{N} \to \{0, 1\}$ by

$$\chi_S(n) = 1 \text{ if } n \in S$$
$$\chi_S(n) = 0 \text{ if } n \notin S$$

Let $f\colon \mathcal{P}(\mathbb{N}) \to \mathbb{N}^{\mathbb{N}}$ be the function defined by $f(S) = \chi_S$. Clearly f is an injection and so $\mathcal{P}(\mathbb{N}) \preceq \mathbb{N}^{\mathbb{N}}$.

Now we define a function $g\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ by

$$g(\pi) = \{2^{\pi(0)+1}, 3^{\pi(1)+1}, \ldots, p_n^{\pi(n)+1}, \ldots\}$$ □

where p_n is the n^{th} prime. Clearly g is an injection. Hence $\mathbb{N}^{\mathbb{N}} \preceq \mathcal{P}(\mathbb{N})$.

By Cantor-Bernstein, $\mathcal{P}(\mathbb{N}) \sim \mathbb{N}^{\mathbb{N}}$.

Heuristic Principle Let S be an infinite set.

1. If each $s \in S$ is determined by a *finite* amount of data, then S is countable.
2. If each $s \in S$ is determined by *infinitely many independent* pieces of data, then S is uncountable.

Definition 5.14. A function $f : \mathbb{N} \to \mathbb{N}$ is *eventually constant* iff there exists $a, b \in \mathbb{N}$ such that

$$f(n) = b \text{ for all } n \geq a.$$

$\text{EC}(\mathbb{N}) = \{ f \in \mathbb{N}^{\mathbb{N}} \mid f \text{ is eventually constant } \}$.

Theorem 5.15. *$\mathbb{N} \sim \text{EC}(\mathbb{N})$.*

Proof. For each $n \in \mathbb{N}$, let $c_n : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$c_n(t) = n \text{ for all } t \in \mathbb{N}.$$

Then we can define an injection $f : \mathbb{N} \to \text{EC}(\mathbb{N})$ by $f(n) = c_n$. Hence $\mathbb{N} \preceq \text{EC}(\mathbb{N})$.

Next we define a function $g : \text{EC}(\mathbb{N}) \to \mathbb{N}$ as follows. Let $\pi \in \text{EC}(\mathbb{N})$. Let $a, b \in \mathbb{N}$ be chosen so that:

1. $\pi(n) = b$ for all $n \geq a$
2. a is the least such integer.

Then

$$g(\pi) = 2^{\pi(0)+1} 3^{\pi(1)+1} \dots p_a^{\pi(a)+1}$$

where p_i is the i^{th} prime. Clearly g is an injection. Thus $\text{EC}(\mathbb{N}) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \text{EC}(\mathbb{N})$. □

6 The proof of Cantor-Berstein

Next we turn to the proof of the Cantor-Bernstein Theorem. We shall make use of the following result.

Definition 6.1. If $f : A \to B$ and $C \subseteq A$, then

$$f[C] = \{ f(c) \mid c \in C \}.$$

Lemma 6.2. *If $f : A \to B$ is an injection and $C \subseteq A$, then*

$$f[A \smallsetminus C] = f[A] \smallsetminus f[C].$$

Proof. Suppose that $x \in f[A \setminus C]$. Then there exists $a \in A \setminus C$ such that $f(a) = x$. In particular $x \in f[A]$. Suppose that $x \in f[C]$. Then there exists $c \in C$ such that $f(c) = x$. But $a \neq c$ and so this contradicts the fact that f is an injection. Hence $x \notin f[C]$ and so $x \in f[A] \setminus f[C]$.

Conversely suppose that $x \in f[A] \setminus f[C]$. Since $x \in f[A]$, there exists $a \in A$ such that $f(a) = x$. Since $x \notin f[C]$, it follows that $a \notin C$. Thus $a \in A \setminus C$ and $x = f(a) \in f[A \setminus C]$. □

Theorem 6.3. *(Cantor-Bernstein) If $A \preceq B$ and $B \preceq A$, then $A \sim B$.*

Proof. Since $A \preceq B$ and $B \preceq A$, there exists injections $f\colon A \to B$ and $g\colon B \to A$. Let $C = g[B] = \{g(b) \mid b \in B\}$.

Claim 6.4. $B \sim C$.

Proof of Claim 6.4. The map $b \mapsto g(b)$ is a bijection from B to C. □

Thus it is enough to prove that $A \sim C$. For then, $A \sim C$ and $C \sim B$, which implies that $A \sim B$.

Let $h = g \circ f\colon A \to C$. Then h is an injection.

Define by induction on $n \geq 0$.

$A_0 = A$ $\qquad$ $C_0 = C$

$A_{n+1} = h[A_n]$ $\qquad$ $C_{n+1} = h[C_n]$

Define $k\colon A \to C$ by $k(x) =$

$= h(x)$ if $x \in A_n \setminus C_n$ for some n

$= x$ otherwise

Claim 6.5. k is an injection.

Proof of Claim 6.5. Suppose that $x \neq x'$ are distinct elements of A. We consider three cases.

Case 1:

Suppose that $x \in A_n \setminus C_n$ and $x' \in A_m \setminus C_m$ for some n, m. Since h is an injection,

$k(x) = h(x) = x \neq x' = h(x) = k(x)$.

Case 2:

Suppose that $x \notin A_n \setminus C_n$ for all n and that $x' \notin A_n \setminus C_n$ for all n. Then

$k(x) = x \neq x' = k(x)$.

Case 3:

Suppose that $x \in A_n \setminus C_n$ and $x' \notin A_m \setminus C_m$ for all m. Then

$k(x) = h(x) \in h[A_n \setminus C_n]$

and

$$h[A_n \setminus C_n] = h[A_n] \setminus h[C_n] = A_{n+1} \setminus C_{n+1}.$$

Hence $k(x) = h(x) \neq x' = k(x')$. □

Claim 6.6. k is a surjection.

Proof of Claim 6.6. Let $x \in C$. There are two cases to consider.

Case 1:
Suppose that $x \notin A_n \smallsetminus C_n$ for all n. Then $k(x) = x$.

Case 2:
Suppose that $x \in A_n \smallsetminus C_n$. Since $x \in C$, we must have that $n = m + 1$ for some m. Since

$$h[A_m \smallsetminus C_m] = A_n \smallsetminus C_n,$$

there exists $y \in A_m \smallsetminus C_m$ such that $k(y) = h(y) = x$. □

This completes the proof of the Cantor-Bernstein Theorem. □

Theorem 6.7. $\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$

Proof. Since $(0, 1) \sim \mathbb{R}$, it follows that $(0, 1) \times (0, 1) \sim \mathbb{R} \times \mathbb{R}$. Hence it is enough to prove that $(0, 1) \sim (0, 1) \times (0, 1)$.

First define $f : (0, 1) \to (0, 1) \times (0, 1)$ by $f(r) = \langle r, r \rangle$. Clearly f is an injection and so $(0, 1) \preceq (0, 1) \times (0, 1)$.

Next define $g : (0, 1) \times (0, 1) \to (0, 1)$ as follows. Suppose that $r, s \in (0, 1)$ have decimal expansions

$$r = 0.r_0 r_1 \dots r_n \dots$$
$$s = 0.s_0 s_1 \dots s_n \dots$$

Then

$$g(\langle r, s \rangle) = 0.r_0 s_0 r_1 s_1 \dots r_n s_n \dots$$

Clearly g is an injection and so $(0, 1) \times (0, 1) \preceq (0, 1)$.

By Cantor-Bernstein, $(0, 1) \sim (0, 1) \times (0, 1)$. □

Exercise 6.8. $\mathbb{R} \smallsetminus \mathbb{N} \sim \mathbb{R}$

Exercise 6.9. $\mathbb{R} \smallsetminus \mathbb{Q} \sim \mathbb{R}$

Exercise 6.10. Let $\mathrm{Sym}(\mathbb{N}) = \{ f \mid f : \mathbb{N} \to \mathbb{N} \text{ is a bijection } \}$. Prove that $\mathcal{P}(\mathbb{N}) \sim \mathrm{Sym}(\mathbb{N})$.

Definition 6.11. Let A be any set. Then a *finite sequence* of elements of A is an object

$$\langle a_0, a_1, \dots, a_n \rangle, \; n \geq 0$$

so that each $a_i \in A$, chosen so that

$$\langle a_0, a_1, \dots, a_n \rangle = \langle b_0, b_1, \dots, b_n \rangle$$

iff $n = m$ and $a_i = b_i$ for $0 \leq i \leq n$.

FinSeq(A) is the set of all finite sequences of elements of A.

Theorem 6.12. *If A is a nonempty countable set, then* $\mathbb{N} \sim \operatorname{FinSeq}(A)$.

Proof. First we prove that $\mathbb{N} \preceq \operatorname{FinSeq}(A)$. Fix some $a \in A$. Then we define $f \colon \mathbb{N} \to \operatorname{FinSeq}(A)$ by

$$f(n) = \langle \underbrace{a, a, a, a, a, \ldots, a}_{n+1 \text{ times}} \rangle.$$

Clearly f is an injection and so $\mathbb{N} \preceq \operatorname{FinSeq}(A)$.

Next we prove that $\operatorname{FinSeq}(A) \preceq \mathbb{N}$. Since A is countable, there exists an injection $e \colon A \to \mathbb{N}$. Define $g \colon \operatorname{FinSeq}(A) \to \mathbb{N}$ by

$$g(\langle a_0, a_1, \ldots, a_n \rangle) = 2^{e(a_0)+1} \ldots p_n^{e(a_n)+1}$$

where p_i is the n^{th} prime. Clearly g is an injection. Hence $\operatorname{FinSeq}(A) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \operatorname{FinSeq}(A)$. □