7 Binary relations

Definition 7.1. A *binary relation* on a set A is a subset $R \subseteq A \times A$. We usually write aRb instead of writing $\langle a, b\rangle \in R$.

Example 7.2. 1. The order relation on $\mathbb{N}$ is given by

$$\{\langle n, m\rangle \mid n, m \in \mathbb{N},\ n < m\}.$$

2. The division relation D on $\mathbb{N} \smallsetminus \{0\}$ is given by

$$D = \{\langle n, m\rangle \mid n, m \in \mathbb{N},\ n \text{ divides } m\}.$$

Observation Thus $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ is the collection of all binary relations on $\mathbb{N}$. Clearly $\mathcal{P}(\mathbb{N}\times\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$ and so $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ is uncountable.

Definition 7.3. Let R be a binary relation on A.

1. R is *reflexive* iff xRx for all $x \in A$.
2. R is *symmetric* iff xRy implies yRx for all $x, y \in A$.
3. R is *transitive* iff xRy and yRz implies xRz for all $x, y, z \in A$.

R is an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Example 7.4. Define the relation R on $\mathbb{Z}$ by

$$aRb \text{ iff } 3 | a - b.$$

Proposition 7.5. *R is an equivalence relation.*

Exercise 7.6. Let $A = \{\langle a, b\rangle \mid a, b \in \mathbb{Z},\ b \neq 0\}$. Define the relation S on A by

$$\langle a, b\rangle S \langle c, d\rangle \text{ iff } ad - bc = 0.$$

Prove that S is an equivalence relation.

Definition 7.7. Let R be an equivalence relation on A. For each $x \in A$, the *equivalence class* of x is

$$[x] = \{y \in A \mid xRy\}.$$

Example 7.4 Cont. The distinct equivalence classes are

$$[0] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$[1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$[2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

Definition 7.8. Let A be a nonempty set. Then $\{B_i \mid i \in I\}$ is a *partition* of A iff the following conditions hold:

1. $\emptyset \neq B_i$ for all $i \in I$.
2. If $i \neq j \in I$, then $B_i \cap B_j = \emptyset$.
3. $A = \bigcup_{i \in I} B_i$.

Theorem 7.9. *Let R be an equivalence relation on A.*

1. *If $a \in A$ then $a \in [a]$.*
2. *If $a, b \in A$ and $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.*

Hence the set of distinct equivalence classes forms a partition of A.

Proof. 1. Let $a \in A$. Since R is reflexive, aRa and so $a \in [a]$.

2. Suppose that $c \in [a] \cap [b]$. Then aRc and bRc. Since R is symmetric, cRb. Since R is transitive, aRb. We claim that $[b] \subseteq [a]$. To see this, suppose that $d \in [b]$. Then bRd. Since aRb and bRd, it follows that aRd. Thus $d \in [a]$. Similarly, $[a] \subseteq [b]$ and so $[a] = [b]$.

□

Theorem 7.10. *Let $\{B_i \mid i \in I\}$ be a partition of A. Define a binary relation R on A by*

aRb iff there exists $i \in I$ such that $a, b \in B_i$.

Then R is an equivalence relation whose equivalence classes are precisely $\{B_i \mid i \in I\}$. □

Example 7.11. How many equivalence relations can be defined on $A = \{1, 2, 3\}$?

Sol'n This is the same as asking how many partitions of A exist.

$\{\{1, 2, 3\}\}$,
$\{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}$,
$\{\{1\}, \{2\}, \{3\}\}$

Hence there are 5 equivalence relations on $\{1, 2, 3\}$.

Exercise 7.12. How many equivalence relations can be defined on $A = \{1, 2, 3, 4\}$?

Challenge Let $\mathrm{EQ}(\mathbb{N})$ be the collection of equivalence relations on $\mathbb{N}$. Prove that $\mathrm{EQ}(\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$.

8 Linear orders

Definition 8.1. Let R be a binary relation on A.

1. R is *irreflexive* iff $\langle a, a\rangle \notin R$ for all $a \in A$.

2. R satisfies the *trichotomy property* iff for all $a, b \in A$, exactly one of the following holds:

   aRb, $a = b$, bRa.

$\langle A, R\rangle$ is a *linear order* iff R is irreflexive, transitive, and satisfies the trichotomy property.

Example 8.2. Each of the following are linear orders.

1. $\langle \mathbb{N}, <\rangle$

2. $\langle \mathbb{N}, >\rangle$

3. $\langle \mathbb{Z}, <\rangle$

4. $\langle \mathbb{Q}, <\rangle$

5. $\langle \mathbb{R}, <\rangle$

Definition 8.3. Let R be a binary relation on A. Then $\langle A, R\rangle$ is a *partial order* iff R is irreflexive and transitive.

Example 8.4. Each of the follow are partial orders, but *not* linear orders.

1. Let A be any nonempty set containing at least two elements. Then $\langle \mathcal{P}(A), \subset\rangle$ is a partial order.

2. Let D be the divisability relation on $\mathbb{N}^+ = \mathbb{N} \smallsetminus \{0\}$. Then $\langle \mathbb{N}^+, D\rangle$ is a partial order.

Definition 8.5. Let $\langle A, <\rangle$ and $\langle B, <\rangle$ be partial orders. A map $f\colon A \to B$ is an *isomorphism* iff the following conditions are satisfied.

1. f is a bijection

2. For all $x, y \in A$, $x < y$ iff $f(x) < f(y)$.

In this case, we say that $\langle A, <\rangle$ and $\langle B, <\rangle$ are isomorphic and write $\langle A, <\rangle \cong \langle B, <\rangle$.

Example 8.6. $\langle \mathbb{Z}, <\rangle \cong \langle \mathbb{Z}, >\rangle$

Proof. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f(x) = -x$. Clearly f is a bijection. Also, if $x, y \in \mathbb{Z}$, then $x < y$
iff $-x > -y$
iff $f(x) > f(y)$.
Thus f is an isomorphism. □

Example 8.7. $\langle \mathbb{N}, < \rangle \ncong \langle \mathbb{Z}, < \rangle$.

Proof. Suppose that $f\colon \mathbb{N} \to \mathbb{Z}$ is an isomorphism. Let $f(0) = z$. Since f is a bijection, there exists $n \in \mathbb{N}$ such that $f(n) = z - 1$. But then $n > 0$ and $f(n) < f(0)$, which is a contradiction. □

Exercise 8.8. Prove that $\langle \mathbb{Z}, < \rangle \ncong \langle \mathbb{Q}, < \rangle$.

Example 8.9. $\langle \mathbb{Q}, < \rangle \ncong \langle \mathbb{R}, < \rangle$.

Proof. Since $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable, there does not exist a bijection $f\colon \mathbb{Q} \to \mathbb{R}$. Hence there does not exist an isomorphism $f\colon \mathbb{Q} \to \mathbb{R}$. □

Example 8.10. $\langle \mathbb{R}, < \rangle \ncong \langle \mathbb{R} \smallsetminus \{0\}, < \rangle$.

Proof. Suppose that $f\colon \mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$ is an isomorphism. For each $n \geq 1$, let $r_n = f(1/n)$. Then

$$r_1 > r_2 > \ldots > r_n > \ldots > f(-1).$$

Let s be the greatest lower bound of $\{r_n \mid n \geq 1\}$. Then there exists $t \in \mathbb{R} \smallsetminus \{0\}$ such that $f(t) = s$. Clearly $t < 0$. Hence $f(t/2) > s$. But then there exists $n \geq 1$ such that $r_n < f(t/2)$. But this means that $t/2 < 1/n$ and $f(t/2) > f(1/n)$, which is a contradiction. □

Question 8.11. Is $\langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \smallsetminus \{0\}, < \rangle$?

Definition 8.12. For each prime p,

$$\mathbb{Z}[1/p] = \{a/p^n \mid a \in \mathbb{Z},\ n \in \mathbb{N}\}.$$

Question 8.13. Is $\langle \mathbb{Z}[1/2], < \rangle \cong \langle \mathbb{Z}[1/3], < \rangle$?

Definition 8.14. A linear order $\langle D, < \rangle$ is a *dense linear order without endpoints* or DLO iff the following conditions hold.

1. For all $a, b \in D$, if $a < b$, then there exists $c \in D$ such that $a < c < b$.
2. For all $a \in D$, there exists $b \in D$ such that $a < b$.
3. For all $a \in D$, there exists $b \in D$ such that $b < a$.

Example 8.15. The following are DLOs.

1. $\langle \mathbb{Q}, < \rangle$
2. $\langle \mathbb{R}, < \rangle$
3. $\langle \mathbb{Q} \setminus \{0\}, < \rangle$
4. $\langle \mathbb{R} \setminus \{0\}, < \rangle$

Theorem 8.16. *For each prime p, $\langle \mathbb{Z}[1/p], < \rangle$ is a DLO.*

Proof. Clearly $\langle \mathbb{Z}[1/p], < \rangle$ linear order without endpoints. Hence it is enough to show that $\mathbb{Z}[1/p]$ is dense. Suppose that $a, b \in \mathbb{Z}[1/p]$. Then there exists $c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a = c/p^n$ and $b = d/p^n$. Clearly $a < a + (1/p^n) \leq b$. Consider

$$r = \frac{c}{p^n} + \frac{1}{p^n} = \frac{cp+1}{p^{n+1}} \in \mathbb{Z}[1/p].$$

Then $a < r < b$. □

Theorem 8.17. *If $\langle A, < \rangle$ and $\langle B, < \rangle$ are countable DLOs then $\langle A, < \rangle \cong \langle B, < \rangle$.*

Corollary 8.18. $\langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \setminus \{0\}, < \rangle$. □

Corollary 8.19. $\langle \mathbb{Z}[1/2], < \rangle \cong \langle \mathbb{Z}[1/3], < \rangle$. □

Corollary 8.20. *If p is any prime, then $\langle \mathbb{Z}[1/p], < \rangle \cong \langle \mathbb{Q}, < \rangle$.* □

Proof of Theorem 8.17. Let $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$. First define $A_0 = \{a_0\}$ and $B_0 = \{b_0\}$ and let $f_0 \colon A_0 \to B_0$ be the map defined by $f_0(a_0) = b_0$.

Now suppose inductively that we have defined a function $f_n \colon A_n \to B_n$ such that the following conditions are satisfied.

1. $\{a_0, \ldots, a_n\} \subseteq A_n \subseteq A$.
2. $\{b_0, \ldots, b_n\} \subseteq B_n \subseteq B$.
3. $f_n \colon A_n \to B_n$ is an order preserving bijection.

We now extend f_n to a suitable function f_{n+1}.

Step 1 If $a_{n+1} \in A_n$, then let $A'_n = A_n$, $B'_n = B_n$, and $f'_n = f_n$. Otherwise, suppose for example that

$$c_0 < c_1 < \ldots < c_i < a_{n+1} < c_{i+1} < \ldots < c_m$$

where $A_n = \{c_0, \ldots, c_m\}$. Choose some element $b \in B$ such that $f_n(c_i) < b < f_n(c_{i+1})$ and define

$$A'_n = A_n \cup \{a_{n+1}\}$$
$$B'_n = B_n \cup \{b\}$$
$$f'_n = f_n \cup \{\langle a_{n+1}, b \rangle\}$$

Step 2 If $b_{n+1} \in B'_n$, then let $A_{n+1} = A'_n$, $B_{n+1} = B'_n$, and $f_{n+1} = f'_n$. Otherwise, suppose for example that

$$d_0 < d_1 < \ldots < d_j < b_{n+1} < d_{j+1} < \ldots < d_t$$

where $B'_n = \{d_0, \ldots, d_t\}$. Choose some element $a \in A$ such that $(f'_n)^{-1}(d_j) < a < (f'_n)^{-1}(d_{j+1})$ and define

$$A_{n+1} = A'_n \cup \{a\}$$
$$B_{n+1} = B'_n \cup \{b_{n+1}\}$$
$$f_{n+1} = f'_n \cup \{\langle a, b_{n+1}\rangle\}.$$

Finally, let $f = \bigcup_{n \geq 0} f_n$. Then $f \colon A \to B$ is an isomorphism. □

9 Extensions

Definition 9.1. Suppose that R, S are binary relations on A. Then S *extends* R iff $R \subseteq R$.

Example 9.2. Consider the binary relations R, S on $\mathbb{N}$ defined by

$$R = \{\langle n, m\rangle \mid n < m\}$$
$$S = \{\langle n, m\rangle \mid n \leq m\}$$

Then S extends R.

Example 9.3. Consider the partial order $\prec$ on $\{a, b, c, d, e\}$ which is

$$\{\langle d, b\rangle, \langle d, a\rangle, \langle d, e\rangle, \langle d, c\rangle, \langle a, b\rangle, \langle e, b\rangle, \langle c, b\rangle\}.$$

Then we can extend $\prec$ to the linear order $<$ defined by the transitive closure of

$$d < e < c < a < b.$$

Exercise 9.4. If $\langle A, \prec\rangle$ is a finite partial order, then we can extend $\prec$ to a linear ordering $<$ of A.

Question 9.5. Does the analogous result hold if $\langle A, \prec\rangle$ is a infinite partial order?

Definition 9.6. If A is a set and $n \geq 1$, then

$$A^n = \{\langle a_1, \ldots, a_n\rangle \mid a_1, \ldots, a_n \in A\}.$$

An *n-ary relation* on A is a subset $R \subseteq A^n$.
An *n-ary operation* on A is a function $f \colon A^n \to A$.