10 Propositional logic

"The study of how the truth value of compound statements depends on those of simple statements."

A reminder of truth-tables.

and $\wedge$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

or $\vee$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

not $\neg$

A	$\neg A$
T	F
F	T

material implication $\rightarrow$

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

iff $\leftrightarrow$

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Now our study actually begins... First we introduce our *formal language*.

Definition 10.1. The *alphabet* consists of the following symbols:

1. the sentence connectives

   $$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

2. the punctuation symbols

   $$(,)$$

3. the sentence symbols

   $$A_1, A_2, \ldots, A_n, \ldots,\ n \geq 1$$

Remark 10.2. Clearly the alphabet is countable.

Definition 10.3. An *expression* is a finite sequence of symbols from the alphabet.

Example 10.4. The following are expressions:
$(A_1 \wedge A_2)$, $((\neg \rightarrow ())A_3$

Remark 10.5. Clearly the set of expressions is countable.

Definition 10.6. The set of *well-formed formulas* (wffs) is defined recursively as follows:

1. Every sentence symbol A_n is a wff.

2. If α and β are wffs, then so are

   $$(\neg\alpha),\ (\alpha\wedge\beta),\ (\alpha\vee\beta),\ (\alpha\rightarrow\beta),\ (\alpha\leftrightarrow\beta)$$

3. No expression is a wff unless it is compelled to be so by repeated applications of (1) and (2).

Remark 10.7.

1. From now on we omit clause (3) in any further recursive definitions.

2. Clearly the set of wffs is countably infinite.

3. Because the definition of a wff is recursive, most of the properties of wffs are proved by induction on the length of a wff.

Example 10.8.

1. $(A_1 \rightarrow (\neg A_2))$ is a wff.

2. $((A_1 \wedge A_2)$ is not a wff. How can we prove this?

Proposition 10.9. *If α is a wff, then α has the same number of left and right parentheses.*

Proof. We argue by induction on the length $n \geq 1$ of the wff α. First suppose that $n = 1$. Then α must be a sentence symbol, say A_n. Clearly the result holds in this case.

Now suppose that $n > 1$ and that the result holds for all wffs of length less than n. Then α must have one of the following forms:

$(\neg\beta)$, $(\beta\wedge\gamma)$, $(\beta\vee\gamma)$, $(\beta\rightarrow\gamma)$, $(\beta\leftrightarrow\gamma)$

for some wffs β, γ of length less than n. By induction hypothesis the result holds for both β and γ. Hence the result also holds for α. □

Definition 10.10. $\mathcal{L}$ is the set of sentence symbols. $\bar{\mathcal{L}}$ is the set of wffs. $\{T, F\}$ is the set of truth values.

Definition 10.11. A *truth assignment* is a function $v \colon \mathcal{L} \to \{T, F\}$.

Definition 10.12. Let v be a truth assignment. Then we define the extension $\bar{v} \colon \bar{\mathcal{L}} \to \{T, F\}$ recursively as follows.

0. If $A_n \in \mathcal{L}$ then $\bar{v}(A_n) = v(A_n)$.

For any $\alpha, \beta \in \bar{\mathcal{L}}$

1. $\bar{v}((\neg\alpha)) =$
   $= T$ if $\bar{v}(\alpha) = F$
   $= F$ otherwise
2. $\bar{v}((\alpha\wedge\beta)) =$
   $= T$ if $\bar{v}(\alpha) = \bar{v}(\beta) = T$
   $= F$ otherwise
3. $\bar{v}((\alpha\vee\beta)) =$
   $= F$ if $\bar{v}(\alpha) = \bar{v}(\beta) = F$
   $= T$ otherwise
4. $\bar{v}((\alpha\rightarrow\beta)) =$
   $= F$ if $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$
   $= T$ otherwise
5. $\bar{v}((\alpha\leftrightarrow\beta)) =$
   $= T$ if $\bar{v}(\alpha) = \bar{v}(\beta)$
   $= F$ otherwise

Possible problem. Suppose there exists a wff α such that α has both the forms $(\beta\rightarrow\gamma)$ and $(\sigma\wedge\varphi)$ for some wffs $\beta, \gamma, \sigma, \varphi$. Then there will be two (possibly conflicting) clauses which define $\bar{v}(\alpha)$.

Fortunately no such α exists...

Theorem 10.13 (Unique readability). *If α is a wff of length greater than 1, then there exists eactly one way of expressing α in the form:*

$(\neg\beta)$, $(\beta\wedge\gamma)$, $(\beta\vee\gamma)$, $(\beta\rightarrow\gamma)$, *or* $(\beta\leftrightarrow\gamma)$

for some shorter wffs β, γ.

We shall make use of the following result.

Lemma 10.14. *Any proper initial segment of a wff contains more left parentheses than right parentheses. Thus no proper initial segment of a wff is a wff.*

Proof. We argue by induction on the length $n \geq 1$ of the wff α. First suppose that $n = 1$. Then α is a sentence symbol, say A_n. Since A_n has no proper initial segments, the result holds vacuously.

Now suppose that $n > 1$ and that the result holds of all wffs of length less than n. Then α must have the form

$$(\neg\beta),\ (\beta\wedge\gamma),\ (\beta\vee\gamma),\ (\beta\rightarrow\gamma), \text{ or } (\beta\leftrightarrow\gamma)$$

for some shorter wffs β and γ. By induction hypothesis, the result holds for both β and γ. We just consider the case when α is $(\beta\wedge\gamma)$. (The other cases are similar.) The proper initial segments of α are:

1. (
2. $(\beta_0$ where β_0 is an initial segment of β
3. $(\beta\wedge$
4. $(\beta\wedge\gamma_0$ where γ_0 is an initial segment of γ.

Using the induction hypothesis and the previous proposition (Proposition 10.9), we see that the result also holds for α. □

Proof of Theorem 10.13. Suppose, for example, that

$$\alpha = (\beta\wedge\gamma) = (\sigma\wedge\varphi).$$

Deleting the first (we obtain that

$$\beta\wedge\gamma) = \sigma\wedge\varphi).$$

Suppose that $\beta \neq \sigma$. Then wlog β is a proper initial segment of σ. But then β isn't a wff, which is a contradiction. Hence $\beta = \sigma$. Deleting β and σ, we obtain that

$$\wedge\gamma) = \wedge\varphi)$$

and so $\gamma = \varphi$.

Next suppose that

$$\alpha = (\beta\wedge\gamma) = (\sigma\rightarrow\varphi).$$

Arguing as above, we find that $\beta = \sigma$ and so

$$\wedge\gamma) = \rightarrow\varphi)$$

which is a contradiction.

The other cases are similar. □

Definition 10.15. Let $v\colon \mathcal{L} \rightarrow \{T, F\}$ be a truth assignment.

1. If φ is a wff, then v *satisfies* φ iff $\bar{v}(\varphi) = T$.
2. If Σ is a set of wffs, then v *satisfies* Σ iff $\bar{v}(\sigma) = T$ for all $\sigma \in \Sigma$.

3. Σ is *satisfiable* iff there exists a truth assignment v which satisfies Σ.

Example 10.16. 1. Suppose that $v\colon \mathcal{L} \to \{T, F\}$ is a truth assignment and that $v(A_1) = F$ and $v(A_2) = T$. Then v satisfies $(A_1 \to A_2)$.

2. $\Sigma = \{A_1, (\neg A_2), (A_1 \to A_2)\}$ is *not* satisfiable.

Exercise 10.17. Suppose that φ is a wff and v_1, v_2 are truth assignments which agree on all sentence symbols appearing in φ. Then $\bar{v_1}(\varphi) = \bar{v_2}(\varphi)$. (*Hint:* argue by induction on the length of φ.)

Definition 10.18. Let Σ be a set of wffs and let φ be a wff. Then Σ *tautologically implies* φ, written $\Sigma \models \varphi$, iff every truth assignment which satisfies Σ also satisfies φ.

Important Observation. Thus $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg\varphi\}$ is not satisfiable.

Example 10.19. $\{A_1, (A_1 \to A_2)\} \models A_2$.

Definition 10.20. The wffs φ, ψ are *tautologically equivalent* iff both $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Example 10.21. $(A_1 \to A_2)$ and $((\neg A_2) \to (\neg A_1)$ are tautologicaly equivalent.

Exercise 10.22. Let σ, τ be wffs. Then the following statements are equivalent.

1. σ and τ are tautologically equivalent.

2. $(\sigma \leftrightarrow \tau)$ is a tautology.

(Hint: do *not* argue by induction on the lengths of the wffs.)