11 The compactness theorem

Question 11.1. Suppose that Σ is an infinite set of wffs and that $\Sigma \models \tau$. Does there necessarily exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$?

A positive answer follows from the following result...

Theorem 11.2 (The Compactness Theorem). *Let Σ be a set of wffs. If every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable, then Σ is satisfiable.*

Definition 11.3. A set Σ of wffs is *finitely satisfiable* iff every finite subset $\Sigma_0 \subset \Sigma$ is satisfiable.

Theorem 11.4 (The Compactness Theorem). *If Σ is a finitely satisfiable set of wffs, then Σ is satisfiable.*

Before proving the compactness theorem, we present a number of its applications.

Corollary 11.5. *If $\Sigma \models \tau$, then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.*

Proof. Suppose not. Then for every finite subset $\Sigma_0 \subseteq \Sigma$, we have that $\Sigma_0 \not\models \tau$ and hence $\Sigma_0 \cup \{(\neg\tau)\}$ is satisfiable. Thus $\Sigma \cup \{(\neg\tau)\}$ is finitely satisfiable. By the Compactness Theorem, $\Sigma \cup \{(\neg\tau)\}$ is satisfiable. But this means that $\Sigma \not\models \tau$, which is a contradiction. □

12 A graph-theoretic application

Definition 12.1. Let E be a binary relation on the set V. Then $\Gamma = \langle V, E\rangle$ is a *graph* iff:

1. E is irreflexive; and
2. E is symmetric.

Example 12.2. Let $V = \{0, 1, 2, 3, 4\}$ and let $E = \{\langle i, j\rangle \mid j = i + 1 \mod 5\}$. This is called the *cycle* of length five.

Definition 12.3. Let $k \geq 1$. Then the graph $\Gamma = \langle V, E\rangle$ is *k-colorable* iff there exists a function $\chi\colon V \to \{1, 2, \dots k\}$. such that for all $a, b \in V$,

(*) if aEb, then $\chi(a) \neq \chi(b)$.

Example 12.4. Any cycle of even length is two-colorable. Any cycle of odd length is three-colorable but not two-colorable.

Theorem 12.5 (Erdös). *A countable graph $\Gamma = \langle V, E\rangle$ is k-colorable iff every finite subgraph $\gamma_0 \subset \Gamma$ is k-colorable.*

Proof. ⇒ Suppose that Γ is k-colorable and let $\chi\colon V \to \{1, 2, \ldots, k\}$ is any k-coloring. Let $\Gamma_0 = \langle V_0, E_0\rangle$ be any finite subgraph of Γ. Then $\chi_0 = \chi|V_0$ is a k-coloring of Γ_0.

⇐ In this direction we use the Compactness Theorem.

Step 1 We choose a suitable propositional language. The idea is to have a sentence symbol for every decision we must make. So our language has sentence symbols:

$C_{v,i}$ for each $v \in V$, $1 \le i \le k$.

(The intended meaning of $C_{v,i}$ is: "color vertex v with color i.")

Step 2 We write down a suitable set of wffs which imposes a suitable set of constraints on our truth assignments. Let Σ be the set of wffs of the following forms:

(a) $C_{v,1} \vee C_{v,2} \vee \ldots \vee C_{v,k}$ for each $v \in V$.

(b) $\neg(C_{v,i} \wedge C_{v,j})$ for each $v \in V$ and $1 \le i \ne j \le k$.

(c) $\neg(C_{v,i} \wedge C_{w,i})$ for each pair $v, w \in V$ of adjacent vertices and each $1 \le i \le k$.

Step 3 We check that we have chosen a suitable set of wffs.

Claim 12.6. Suppose that υ is a truth assignment which satisfies Σ. Then we can define a k-coloring $\chi\colon \Gamma \to \{1, \ldots, k\}$ by

$\chi(v) = i$ iff $\upsilon(C_{v,i}) = T$.

Proof. By (a) and by (b), for each $v \in V$, there exists a unique $1 \le i \le k$ such that $\upsilon(C_{v,i}) = T$. Thus $\chi\colon V \to \{1, \ldots\}$ is a function. By (c), if $v, w \in V$ are adjacent, then $\chi(v) \ne \chi(w)$. Hence χ is a k-coloring. □

Step 4 We next prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $V_0 \subseteq V$ be the finite set of vertices that are mentioned in Σ_0. Then the finite subgraph $\Gamma_0 = \langle V_0, E_0\rangle$ is k-colorable. Let

$$\chi\colon V_0 \to \{1, \ldots, k\}$$

be a k-coloring of Γ_0. Let υ_0 be a truth assignment such that if $v \in V_0$ and $1 \le i \le k$, then

$$\upsilon(C_{v,i}) = T \ \text{ iff } \ \chi_0(v) = i.$$

Clearly υ_0 satisfies Σ_0.

By the Compactness Theorem, Σ is satisfiable. Hence Γ is k-colorable. □

13 Extending partial orders

Theorem 13.1. *Let $\langle A, \prec\rangle$ be a countable partial order. Then there exists a linear ordering $<$ of A which extends $\prec$.*

Proof. We work with the propositional language which has sentence symbols

$$L_{a,b} \quad \text{for} \quad a \neq b \in A$$

Let Σ be the following set of wffs:

(a) $L_{a,b} \vee L_{b,a}$ for $a \neq b \in A$

(b) $\neg(L_{a,b} \wedge L_{b,a})$ for $a \neq b \in A$

(c) $((L_{a,b} \wedge L_{b,c}) \rightarrow L_{a,c})$ for distinct $a, b, c \in A$

(d) $L_{a,b}$ for distinct $a, b \in A$ with $a \prec b$.

Claim 13.2. Suppose that v is a truth assignment which satisfies Σ. Define the binary relation $<$ on A by

$$a < b \text{ iff } v(L_{a,b}) = T.$$

Then $<$ is a linear ordering of A which extends $\prec$.

Proof. Clearly $<$ is irreflexive. By (a) and (b), $<$ has the trichotomy property. By (c), $<$ is transitive. Finally, by (d), $<$ extends $\prec$. □

Next we prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $A_0 \subseteq A$ be the finite set of elements that are mentioned in Σ_0 and consider the partial order $\langle A_0, \prec_0\rangle$. Then there exists a partial ordering $<_0$ of A_0 extending $\prec_0$. Let v_0 be the truth assignment such that if $a \neq b \in A_0$, then

$$v_0(L_{a,b}) = T \text{ iff } a \leq_0 b.$$

Clearly v_0 satisfies Σ_0.

By the compactness theorem, Σ is satisfiable. Hence there exists a linear ordering $<$ of A which extends $\prec$. □

14 Hall's Theorem

Definition 14.1. Suppose that S is a set and that $\langle S_i \mid i \in I\rangle$ is an indexed collection of (not necessarily distinct) subsets of S. A system of *distinct representatives* is a choice of elements $x_i \in S_i$ for $i \in I$ such that if $i \neq j \in I$, then $x_i \neq x_j$.

Example 14.2. Let $S = \mathbb{N}$ and let $\langle S_n \mid n \in \mathbb{N} \rangle$ be defined by

$$S_n = \{n, n+1\}$$

Thus $S_0 = \{0, 1\}$, $S_1 = \{1, 2\}$, ... Then we can take $x_i = i \in S_i$.

Theorem 14.3 (Hall's Matching Theorem (1935)). *Let S be any set and let $n \in \mathbb{N}^+$. Let $\langle S_1, S_2, \ldots, S_n \rangle$ be an indexed collection of subsets of S. Then a necessary and sufficent condition for the existance of a system of distinct representatives is:*

(H) For every $1 \leq k \leq n$ and choice of k distinct indices $1 \leq i_1, \ldots, i_k \leq n$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Challange: Prove this!

Problem 14.4. State and prove an infinite analogue of Hall's Matching Theorem.

First Attempt Let S be any set and let $\langle S_n \mid n \in \mathbb{N}^+ \rangle$ be an indexed collection of subsets of S. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:

(H^*) For every $k \in \mathbb{N}^+$ and choice of k distinct indices $i_1, \ldots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Counterexample Take $S_1 = \mathbb{N}$, $S_2 = \{0\}$, $S_3 = \{1\}$, ..., $S_n = \{n-2\}$, ... Clearly (H^*) is satisfied and yet there is *no* system of distinct representatives.

Question 14.5. Where does the compactness argument break down?

Theorem 14.6 (Infinite Hall's Theorem). *Let S be any set and let $\langle S_n \mid n \in \mathbb{N}^+ \rangle$ be an indexed collection of* finite *subsets of S. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:*

(H^*) For every $k \in \mathbb{N}^+$ and choice of k distinct indices $i_1, \ldots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Proof. We work with the propositional language with sentence symbols

$$C_{n,x}. \quad n \in \mathbb{N}^+, \quad x \in S_n.$$

Let Σ be the following set of wffs:

(a) $\neg(C_{n,x} \wedge C_{m,x})$ for $n \neq m \in \mathbb{N}^+$, $x \in S_n \cap S_m$.

(b) $\neg(C_{n,x} \wedge C_{n,y})$ for $n \in \mathbb{N}^+$, $x \neq y \in S_n \cap S_m$.

(c) $(C_{n,x_1} \vee \ldots \vee C_{n,x_k})$ for $n \in \mathbb{N}^+$, where $S_n = \{x_1, \ldots, x_k\}$.

Claim 14.7. Suppose that υ is a truth assignment which satisfies Σ. Then we can define a system of distinct representatives by

$$x \in S_n \text{ iff } \upsilon(C_{n,x}) = T.$$

Proof. By (b) and (c), each S_n gets assigned a unique representative. By (a), distinct sets $S_m \neq S_m$ get assigned distinct representatives. □

Next we prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $\{i_1, \dots, i_l\}$ be the indices that are mentioned in Σ_0. Then $\{S_{i_1}, \dots S_{i_l}\}$ satisfies condition (H). By Hall's Theorem, there exists a set of distinct representatives for $\{S_{i_1}, \dots S_{i_l}\}$; say, $x_r \in S_{i_r}$. Let υ_0 be the truth assignment such that for $1 \leq r \leq l$ and $x \in S_{i_r}$,

$$\upsilon(C_{i_r,x}) = T \text{ iff } x = x_r.$$

Clearly υ_0 satisfies Σ_0.

By the Compactness Theorem, Σ is satisfiable. Hence there exists a system of distinct representatives. □

15 Proof of compactness

Theorem 15.1 (The Compactness Theorem). *If Σ is a finitely satisfiable set of wffs, then Σ is satisfiable.*

Basic idea Imagine that for each sentence symbol A_n, either $A_n \in \Sigma$ or $\neg A_n \in \Sigma$. Then there is only one possibility for a truth assignment υ which satisfies Σ: namely,

$$\upsilon(A_n) = T \text{ iff } A_n \in \Sigma.$$

Presumably this υ works...

In the general case, we extend Σ to a finitely satisfiable set Δ as above. For technical reasons, we construct Δ so that for *every* wff α, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Lemma 15.2. *Suppose that Σ is a finitely satisfiable set of wffs. If α is any wff, then either $\Sigma \cup \{\alpha\}$ is finitely satisfiable or $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.*

Proof. Suppose that $\Sigma \cup \{\alpha\}$ isn't finitely satisfiable. Then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \cup \{\alpha\}$ isn't satisfiable. Thus $\Sigma \models \neg\alpha$. We claim that $\Sigma \cup \neg\alpha$ is fintely satisfiable. Let $\Delta \subseteq \Sigma \cup \{\neg\alpha\}$ be any finite subset. If $\Delta \subseteq \Sigma$ then Δ is satisfiable. Hence we can suppose that $\Delta = \Delta_0 \cup \{\neg\alpha\}$ for some finite subset $\Delta_0 \subseteq \Sigma$. Since Σ is finitely satisfiable, ther exists a truth assignment υ which satisfies $\Sigma_0 \cap \Delta_0$. Since $\Sigma_0 \models \neg\alpha$, it follows that $\bar{\upsilon}(\neg\alpha) = T$. Hence υ satisfies $\Delta_0 \cup \{\neg\alpha\}$. □

Proof of the Compactness Theorem. Let Σ be a finitely satisfiable set of wffs. Let

$$\alpha_1, \alpha_2, \dots, \alpha_n, \dots \quad n \geq 1$$

be an enumeration of all the wffs $\alpha \in \bar{\mathcal{L}}$. We shall inductively define an increasing sequence of finitely satisfiable sets of wffs

$$\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_n \subseteq \dots$$

First let $\Delta_0 = \Sigma$. Suppose inductively that Δ_n has been defined. Then

$$\begin{aligned} \Delta_{n+1} &= \Delta_n \cup \{\alpha_{n+1}\}, \text{ if this is finitely satisfiable} \\ &= \Delta_n \cup \{(\neg\alpha_{n+1})\}, \text{ otherwise.} \end{aligned}$$

By the lemma, Δ_{n+1} is also finitely satisfiable. Finally define

$$\Delta = \bigcup_n \Delta_n.$$

Claim 15.3. Δ is finitely satisfiable.

Proof. Suppose that $\Phi \subseteq \Delta$ is a finite subset. Then there exists an n such that $\Phi \subseteq \Delta_n$. Since Δ_n is finitely satisfiable, Φ is satisfiable. □

Claim 15.4. If α is any wff, then either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$.

Proof. There exists an $n \geq 1$ such that $\alpha = \alpha_n$. By construction, either $\alpha_n \in \Delta_{n+1}$ or $(\neg\alpha_n) \in \Delta_{n+1}$; and $\Delta_{n+1} \subseteq \Delta$. □

Define a truth assignment $v\colon \mathcal{L} \to \{T, F\}$ by

$$v(A_l) = T \text{ iff } A_l \in \Delta.$$

Claim 15.5. For every wff α, $\bar{v}(\alpha) = T$ iff $\alpha \in \Delta$.

Proof. We argue by induction on the length m of the wff α. First suppose that $m = 1$. Then α is a sentence symbol; say, $\alpha = A_l$. By definition

$$\bar{v}(A_l) = v(A_l) = T \text{ iff } A_l \in \Delta.$$

Now suppose that $m > 1$. Then α has the form

$$(\neg\beta), (\beta\wedge\gamma), (\beta\vee\gamma), (\beta\to\gamma), (\beta\leftrightarrow\gamma)$$

for some shorter wffs β, γ.

Case 1 Suppose that $\alpha = (\neg\beta)$. Then

$$\begin{aligned} \bar{v}(\alpha) = T \quad & \text{iff} \quad \bar{v}(\beta) = F \\ & \text{iff} \quad \beta \notin \Delta \ \text{ by induction hypothesis} \\ & \text{iff} \quad (\neg\beta) \in \Delta \ \text{ by Claim 15.4} \\ & \text{iff} \quad \alpha \in \Delta \end{aligned}$$

Case 2 Suppose that α is $(\beta \vee \gamma)$. First suppose that $\bar{v}(\alpha) = T$. Then $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. By induction hypothesis, $\beta \in \Delta$ or $\gamma \in \Delta$. Since Δ is finitely satisfiable, $\{\beta, (\neg(\beta \vee \gamma))\} \not\subseteq \Delta$ and $\{\gamma, (\neg(\beta \vee \gamma))\} \not\subseteq \Delta$. Hence $(\neg(\beta \vee \gamma)) \notin \Delta$ and so $(\beta \vee \gamma) \in \Delta$.

Conversely suppose that $(\beta \vee \gamma) \in \Delta$. Since Δ is finitely satisfiable, $\{(\neg\beta), (\neg\gamma), (\beta \vee \Gamma)\} \not\subseteq \Delta$. Hence $(\neg\beta) \notin \Delta$ or $(\neg\gamma) \notin \Delta$; and so $\beta \in \Delta$ or $\gamma \in \Delta$. By induction hypothesis, $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. Hence $\bar{v}(\beta \vee \gamma) = T$.

Exercise 15.6. Write out the details for the other cases.

□

Thus v satisfies Δ. Since $\Sigma \subseteq \Delta$, it follows that v satisfies Σ. □