16 Trees and Konig's Lemma

Definition 16.1. A partial order $\langle T, \prec \rangle$ is a *tree* iff the following conditions are satisfied.

1. There exists a unique minimal element $t_0 \in T$ called the *root*.
2. For each $t \in T$, the set
$$\mathrm{Pr}_T(t) = \{ s \in T \mid s \prec t \}$$
is a finite set which is linearly ordered by $\prec$.

Example 16.2. The complete binary tree is defined to be

$$T_2 = \{ f \mid f \colon n \to \{0, 1\} \}$$

ordered by
$$f \prec g \ \text{ iff } \ f \subset g.$$

Definition 16.3. Let $\langle T, \prec \rangle$ be a tree.

1. If $t \in T$, then the *height* of t is defined to be
$$\mathrm{ht}_T(t) = |\mathrm{Pr}_T(t)|.$$
2. For each $n \geq 0$, the n^{th} *level* of T is
$$\mathrm{Lev}_T(n) = \{ t \in T \mid \mathrm{ht}_T(t) = n \}.$$
3. For each $t \in T$, the set of *immediate successors* of t is
$$\mathrm{succ}_T(t) = \{ s \in T \mid t \prec s \text{ and } \mathrm{ht}_T(s) = \mathrm{ht}_T(t) + 1 \}.$$
4. T is *finitely branching* iff each $t \in T$ has a finite (possibly empty) set of immediate successors.
5. A *branch* $\mathcal{B}$ of T is a maximal linearly ordered subset of T.

Example 16.4. Consider the complete binary tree T_2. If $\varphi \colon \mathbb{N} \to \{0, 1\}$, then we can define a corresponding branch of T_2 by

$$\mathcal{B}_\varphi = \{ \varphi | n \mid n \in \mathbb{N} \}.$$

Conversely, let $\mathcal{B}$ be an arbitrary branch of T_2. Let $\varphi = \bigcup \mathcal{B}$. Then $\varphi \colon \mathbb{N} \to \{0, 1\}$ and $\mathcal{B} = \mathcal{B}_\varphi$.

Exercise 16.5. Let $\langle T, \prec \rangle$ be a tree. Then the following are equivalent:

1. T is finitely branching.
2. $\text{Lev}_T(n)$ is finite for all $n \geq 0$.

Lemma 16.6 (König). *Suppose that T is an infinite finitely branching tree. Then there exists an infinite branch $\mathcal{B}$ through T.*

Remark 16.7. Note that such a branch $\mathcal{B}$ necessarily satisfies:

$$|\mathcal{B} \cap \text{Lev}_T(n)| = 1 \quad \text{for all } n \geq 0.$$

First we shall give a proof of König's Lemma, using the Compactness Theorem.

Proof of König's Lemma. Let $\langle T, \prec \rangle$ be an infinite finitely branching tree. Then each level $\text{Lev}_T(n)$ is finite and so T is countably infinite. We shall work with the propositional language with sentence symbols $\{B_t \mid t \in T\}$. Let Σ be the following set of wffs:

(a) $B_{t_1} \vee \ldots \vee B_{t_l}$ where $\text{Lev}_T(n) = \{t_1, \ldots, t_l\}$ and $n \geq 0$.

(b) $\neg(B_{t_i} \wedge B_{t_j})$ where $\text{Lev}_T(n) = \{t_1, \ldots, t_l\}$, $n \geq 0$, and $1 \leq i < j \leq l$.

(c) $(B_s \rightarrow B_t)$ for $s, t \in T$ with $s \prec t$.

Claim 16.8. Suppose that v is a truth assignment which satisfies Σ. Then

$$\mathcal{B} = \{t \in T \mid v(B_t) = T\}$$

is an infinite branch through T.

Proof. By (a) and (b), $\mathcal{B}$ intersects every level in a unique point. Suppose that $s \neq t \in \mathcal{B}$. Then wlog we have that $\text{ht}_T(s) < \text{ht}_T(t)$. Let $n = \text{ht}_T(s)$. By (c), $\mathcal{B}$ must contain the predecessor of t in $\text{Lev}_T(n)$, which must be equal to s. Thus $s \prec t$. It follows that $\mathcal{B}$ is linearly ordered. □

We claim that Σ is finitely satisfiable. Let $\Sigma_0 \subseteq \Sigma$ be a finite subset. Then there exists $n \geq 0$ such that if $t \in T$ is mentioned in Σ_0, then $\text{ht}_T(t) < n$. Choose $t_0 \in \text{Lev}_T(n)$ and let v_0 be the truth assignment such that for all $t \in T$ with $\text{ht}_T(t) < n$,

$$v_0(B_t) = T \quad \text{iff} \quad t < t_0.$$

Clearly v_0 satisfies Σ_0. By Compactness, Σ is satisfiable and hence T has an infinite branch. □

Next we shall give a direct proof of König's Lemma.

Proof of König's Lemma. Let T be an infinite finitely branching tree. We shall define a sequence of elements $t_n \in T$ inductively so that the following conditions are satisfied:

(a) $t_n \in \mathrm{Lev}_T(n)$

(b) If $m < n$ then $t_m \prec t_n$.

(c) $\{s \in T \mid t_n \prec s\}$ is infinite.

First let $t_0 \in \mathrm{Lev}_T(0)$ be the root. Clearly the above conditions are satisfied.. Assume inductively that t_n has been defined. Then t_n has a finite set of immediate successors; say $\{a_1, \ldots, a_l\}$. If $t_n \prec s$ and $\mathrm{ht}_T(s) > n+1$, then there exists $1 \leq i \leq l$ such that $a_i \prec s$. By the pigeon hole principle, there exists $1 \leq i \leq l$ such that a_i satisfies (c). Then we define $t_{n+1} = a_i$. Clearly $\mathcal{B} = \{t_n \mid n \geq 0\}$ is an infinite branch through T. □

Next we present an application of König's Lemma.

Theorem 16.9 (Erdös). *A countably infinite graph Γ is k-colorable iff every finite subgraph of Γ is k-colorable.*

Proof. ($\Rightarrow$) Trival!

($\Leftarrow$) Suppose that every finite subgraph of Γ is k-colorable. Let $\Gamma = \{v_1, v_2, \ldots, v_n, \ldots\}$; and for each $n \geq 1$, let $\Gamma_n = \{v_1, \ldots, v_n\}$ and let C_n be the set of k-colorings of Γ_n. Let T be the tree with levels defined by

$$\begin{aligned} \mathrm{Lev}_T(0) &= \{\emptyset\} \\ \mathrm{Lev}_T(n) &= C_n \quad \text{for } n \geq 0 \end{aligned}$$

partially ordered as follows. Suppose that $\chi \in \mathrm{Lev}_T(n)$ and $\theta \in \mathrm{Lev}_T(m)$ where $1 \leq n < m$. Then

$$\chi \prec \theta \quad \text{iff} \quad \chi = \theta | \{v_1, \ldots, v_n\}.$$

Clearly T is an infinite finitely branching tree. By König's Lemma, there exists an infinite branch $\mathcal{B} = \{\chi_n \mid n \in \mathbb{N}\}$ through T, where $\chi_n \in \mathrm{Lev}_T(n)$. We claim that $\chi = \bigcup_n \chi_n$ is a k-coloring of Γ. It is clear that $\chi \colon \Gamma \to \{1, \ldots, k\}$. Next suppose that $a \neq b \in \Gamma$ are adjacent vertices. Then there exists $n \geq 1$ such that $a, b \in \Gamma_n$. By definition, we have that $\chi(a) = \chi_n(a)$ and $\chi(b) = \chi_n(b)$. Since χ_n is a k-coloring of Γ_n, it follows that $\chi_n(a) \neq \chi_n(b)$. Thus $\chi(a) \neq \chi(b)$. □

Finally we use König's Lemma to give a proof of the Compactness Theorem.

Proof of Compactness Theorem. Suppose that Σ is a finitely satisfiable set of wffs in the propositional language with sentence symbols $\{A_1, A_2, \ldots, A_n, \ldots\}$. We define a tree T as follows.

- $\mathrm{Lev}_T(0) = \{\emptyset\}$

- If $n \geq 1$, then $\text{Lev}_T(n)$ is the set of all partial truth assignments $\upsilon \colon \{A_1, \dots, A_n\} \to \{T, F\}$ which satisfy every $\sigma \in \Sigma$ which only mention $A_1, \dots, A_n$.

We partially order T as follows. Suppose that $\upsilon \in \text{Lev}_T(n)$ and $\upsilon' \in \text{Lev}_T(m)$, where $1 \leq n < m$. Then

$$\upsilon \prec \upsilon' \text{ iff } \upsilon = \upsilon' | \{A_1, \dots, A_n\}.$$

Clearly $|\text{Lev}_T(n)| \leq 2^n$ and so each level $\text{Lev}_T(n)$ is finite.

Claim 16.10. For each $n \geq 0$, $\text{Lev}_T(n) \neq \emptyset$.

Proof. Clearly we can suppose that $n \geq 1$. Let Σ_n be the set of wffs in Σ which only involve $A_1, \dots, A_n$. If Σ_n is finite, the result holds by the finite satisfiability of Σ. Hence we can suppose that Σ_n is infinite; say $\Sigma_n = \{\sigma_1, \sigma_2, \dots, \sigma_t, \dots\}$. For each $t \geq 1$, let $\Delta_t = \{\sigma_1, \dots, \sigma_t\}$. Then there exists a partial truth assignment $\omega_t \colon \{A_1, \dots, A_n\} \to \{T, F\}$ which satisfies Δ_t. By the pigeon hole principle, there exists a *fixed* $\omega \colon \{A_1, \dots, A_n\} \to \{T, F\}$ such that $\omega_t = \omega$ for infinitely many $t \geq 1$. Clearly $\omega \in \text{Lev}_T(n)$. □

Thus T is an infinite, finitely branching tree. By König's Lemma, there exists an infinite branch $\mathcal{B} = \{\upsilon_n \mid n \in \mathbb{N}\}$ through T, where $\upsilon_n \in \text{Lev}_T(n)$. It follows that $\upsilon = \bigcup_n \upsilon_n$ is a truth assignment which satisfies Σ. □