# 17 First Order Logic

**Definition 17.1.** The *alphabet* of a first order language $\mathcal{L}$ consists of:

A. Symbols common to all languages (Logical Symbols)

  (a) Parentheses (, )
  (b) Connectives $\rightarrow, \neg$
  (c) Variables $v_1, v_2, \ldots, v_n, \ldots \; n \geq 0$
  (d) Quantifier $\forall$
  (e) Equality symbol $=$

B. Symbols particular to the language (Non-logical Symbols)

  (a) For each $n \geq 1$, a (possibly empty) countable set of $n$-place predicate symbols.
  (b) A (possibly empty) countable set of constant symbols.
  (c) For each $n \geq 1$, a (possibly empty) countable set of $n$-place function symbols.

**Remark 17.2.** It is easily checked that the alphabet is countable.

**Definition 17.3.** An *expression* is a finite sequence of symbols from the alphabet.

**Remark 17.4.** The set of expressions is countable.

**Definition 17.5.** The set of terms is defined inductively as follows:

1. Each variable and each constant symbol is a term.
2. If $f$ is an $n$-place function symbol and $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a term.

**Definition 17.6.** An *atomic formula* is an expression of the form

$$Pt_1 \ldots t_n$$

where $P$ is an $n$-place predicate symbol and $t_1, \ldots, t_n$ are terms.

**Remark 17.7.** The equality symbol $=$ is a two-place predicate symbol. Hence every language has atomic formulas.

**Definition 17.8.** The set of *well-formed formulas* (wffs) is defined inductively as follows:

1. Every atomic formula is a wff.
2. If $\alpha$ and $\beta$ are wffs and $v$ is a variable, then

$$(\neg\alpha), (\alpha\rightarrow\beta), \quad \text{and} \quad \forall v\alpha$$

   are wffs.

**Some abbreviations** We usually write

| | | |
|---|---|---|
| $(\alpha \vee \beta)$ | instead of | $((\neg\alpha)\rightarrow\beta)$ |
| $(\alpha \wedge \beta)$ | " | $(\neg(\alpha\rightarrow(\neg\beta)))$ |
| $\exists v\alpha$ | " | $(\neg\forall v(\neg\alpha))$ |
| $u = t$ | " | $= ut$ |
| $u \neq t$ | " | $(\neg = ut)$ |

We also use common sense in our use of parentheses.

**Definition 17.9.** Let $x$ be a variable.

1. If $\alpha$ is atomic, then $x$ *occurs free* in $\alpha$ iff $x$ occurs in $\alpha$.
2. $x$ occurs free in $(\neg\alpha)$ iff $x$ occurs free in $\alpha$.
3. $x$ occurs free in $(\alpha\rightarrow\beta)$ iff $x$ occurs free in $\alpha$ or $x$ occurs free in $\beta$.
4. $x$ occurs free in $\forall v\alpha$ iff $x$ occurs free in $\alpha$ and $x \neq v$.

**Definition 17.10.** The wff $\sigma$ is a *sentence* iff $\sigma$ has no free variables.

# 18 Truth and Structures

**Definition 18.1.** A *structure* $\mathcal{A}$ for the first order language $\mathcal{L}$ consists of:

1. a non-empty set $A$, the *universe* of $\mathcal{A}$.
2. for each $n$-place predicate symbol $P$, an $n$-ary relation $P^{\mathcal{A}} \subseteq A^n$.
3. for each constant symbol $c$, an element $c^{\mathcal{A}} \in A$.
4. for each function symbol $f$, an $n$-ary operation $f^{\mathcal{A}}\colon A^n \rightarrow A$.

**Example 18.2.** Suppose that $\mathcal{L}$ has the following non-logical symbols:

1. a 1-place predicate symbol $S$
2. a 2-place predicate symbol $R$
3. a constant symbol $c$
4. a 1-place function symbol $f$.

Then we can define a structure

$$\mathcal{A} = \langle A; S^{\mathcal{A}}, R^{\mathcal{A}}, c^{\mathcal{A}}, f^{\mathcal{A}} \rangle$$

for $\mathcal{L}$ as follows:

1. $A = \{1, 2, 3, 4\}$
2. $S^{\mathcal{A}} = \{\langle 2 \rangle, \langle 3 \rangle\}$
3. $R^{\mathcal{A}} = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle, \langle 4, 1 \rangle\}$
4. $c^{\mathcal{A}} = 1$
5. $f^{\mathcal{A}} \colon A \to A$ where $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, and $4 \mapsto 1$.

**Target** Let $\mathcal{L}$ be any first order language. For each sentence $\sigma$ and each structure $\mathcal{A}$ for $\mathcal{L}$, we want to define

$$\mathcal{A} \models \sigma$$

"$\mathcal{A}$ satisfies $\sigma$" or "$\sigma$ is true in $\mathcal{A}$".

**Example 18.3 (Example Cont.).** Let $\sigma$ be the sentence

$$\forall x \forall y (fx = y \rightarrow Rxy)$$

Clearly

$$\mathcal{A} \models \sigma.$$

First we need to define a more involved notion. Let

- $\varphi$ be a wff
- $\mathcal{A}$ be a structure for $\mathcal{L}$
- $s \colon V \to A$ be a function, where $v$ is the set of variables.

Then we will define

$$\mathcal{A} \models \varphi[s]$$

"$\varphi$ is true in $\mathcal{A}$ if each free occurence of $x$ in $\varphi$ is interpreted as $s(x)$ in $A$."

**Step 1**
Let $T$ be the set of terms. We first define an extension $\bar{s} \colon T \to A$ as follows:

1. For each variable $v \in V$, $\bar{s}(v) = s(v)$.
2. For each constant symbol $c$, $\bar{s}(c) = c^{\mathcal{A}}$.

3. If $f$ is an $n$-place function symbol and $t_1, \dots, t_n$ are terms, then

$$\bar{s}(ft_1 \dots t_n) = f^{\mathcal{A}}(\bar{s}(t_1), \dots, \bar{s}(t_n)).$$

**Step 2** Atomic formulas.

(a). $\mathcal{A} \models \; = t_1 t_2[s] \quad \text{iff} \quad \bar{s}(t_1) = \bar{s}(t_2)$.

(b). If $P$ is an $n$-place predicate symbol different from $=$ and $t_1, \dots, t_n$ are terms, then

$$\mathcal{A} \models Pt_1 \dots t_n[s] \; \text{ iff } \; \langle \bar{s}(t_1), \dots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}}.$$

**Step 3** Other wffs.

(c). $\mathcal{A} \models (\neg\alpha)[s]$ iff $\mathcal{A} \not\models \alpha[s]$.

(d). $\mathcal{A} \models (\alpha \rightarrow \beta)[s]$ iff $\mathcal{A} \not\models \alpha[s]$ or $\mathcal{A} \models \beta[s]$.

(e). $\mathcal{A} \models \forall x \alpha[s]$ iff for all $a \in A$, $\mathcal{A} \models \alpha[s(x|a)]$ where $s(x|a)$ is defined by

$$\begin{aligned} s(x|a)(y) &= s(y), y \neq x \\ &= a, y = x \end{aligned}$$

**Theorem 18.4.** *Assume that $s_1, s_2 \colon V \to A$ agree on all free variables (if any) of the wff $\varphi$. Then*

$$\mathcal{A} \models \varphi[s_1] \; \text{ iff } \; \mathcal{A} \models \varphi[s_2].$$

**Proof slightly delayed.**

**Corollary 18.5.** *If $\sigma$ is a sentence, then either*

1. *$\mathcal{A} \models \sigma[s]$ for all $s \colon V \to A$ or*
2. *$\mathcal{A} \not\models \sigma[s]$ for all $s \colon V \to A$.*

**Definition 18.6.** Let $\sigma$ be a sentence. Then $\mathcal{A} \models \sigma$ iff $\mathcal{A} \models \sigma[s]$ for all $s \colon V \to A$.

**Exercise 18.7.** Let $\mathcal{A}$ be a structure and let $t$ be a term. If $s_1, s_2 \colon V \to A$ agree on all variables (if any) in $t$, then $\bar{s_1}(t) = \bar{s_2}(t)$.

*Proof of Theorem 18.4.* We argue by induction on the complexity of $\varphi$.

**Case 1** Suppose that $\varphi$ is an atomic formula. First suppose that $\varphi$ is $= t_1 t_2$. By the Exercise, $\bar{s_1}(t_1) = \bar{s_2}(t_1)$ and $\bar{s_1}(t_2) = \bar{s_2}(t_2)$. Hence

$$\begin{aligned} \mathcal{A} \models\ = t_1 t_2[s_1] \quad & \text{iff} \quad \bar{s_1}(t_1) = \bar{s_1}(t_2) \\ & \text{iff} \quad \bar{s_2}(t_1) = \bar{s_2}(t_2) \\ & \text{iff} \quad \mathcal{A} \models\ = t_1 t_2[s_2]. \end{aligned}$$

Next suppose that $\varphi$ is $P t_1 \dots t_n$. Again by the Exercise, $\bar{s_1}(t_i) = \bar{s_2}(t_i)$ for $1 \leq i \leq n$. Hence

$$\begin{aligned} \mathcal{A} \models P t_1 \dots t_n[s_1] \quad & \text{iff} \quad \langle \bar{s_1}(t_1), \dots, \bar{s_1}(t_n) \rangle \in P^{\mathcal{A}} \\ & \text{iff} \quad \langle \bar{s_2}(t_1), \dots, \bar{s_2}(t_n) \rangle \in P^{\mathcal{A}} \\ & \text{iff} \quad \mathcal{A} \models P t_1 \dots t_n[s_2]. \end{aligned}$$

**Case 2** Suppose that $\varphi$ is $(\neg\psi)$. Then $s_1$, $s_2$ agree on the free variables of $\psi$. Hence

$$\begin{aligned} \mathcal{A} \models (\neg\psi)[s_1] \quad & \text{iff} \quad \mathcal{A} \not\models \psi[s_1] \\ & \text{iff} \quad \mathcal{A} \not\models \psi[s_2] \text{ by ind. hyp.} \\ & \text{iff} \quad \mathcal{A} \models (\neg\psi)[s_2]. \end{aligned}$$

**Case 3** A similar argument deals with the case when $\varphi$ is $(\psi \rightarrow \theta)$.

**Case 4** Suppose that $\varphi$ is $\forall x \psi$. Then $s_1, s_2$ agree on all free variables of $\psi$ except possibly $x$. Hence for all $a \in A$, $s_1(x|a)$ and $s_2(x|a)$ agree on all free variables of $\psi$. Thus

$$\begin{aligned} \mathcal{A} \models \forall x \psi[s_1] \quad & \text{iff} \quad \text{for all } a \in A,\ \mathcal{A} \models \psi[s_1(x|a)] \\ & \text{iff} \quad \text{for all } a \in A,\ \mathcal{A} \models \psi[s_2(x|a)] \\ & \text{iff} \quad \mathcal{A} \models \forall x \psi[s_2]. \end{aligned}$$

□