19 Compactness in first order logic

Definition 19.1. Let Σ be a set of wffs.

(a) $\mathcal{A}$ *satisfies* Σ with s iff $\mathcal{A} \models \sigma[s]$ for all $\sigma \in \Sigma$.

(b) Σ is *satisfiable* iff there exists a structure $\mathcal{A}$ and a function $s \colon V \to A$ such that $\mathcal{A}$ satisfies Σ with s.

(c) Σ is *finitely satisfiable* iff every finite subset of Σ is satisfiable.

One of the deepest results of the course:

Theorem 19.2 (Compactness). *Let Σ be a set of wffs in the first order language $\mathcal{L}$. If Σ is finitely satisfiable, then Σ is satisfiable.*

Application of the Compactness Theorem Let $\mathcal{L}$ be the language of arithmetic; ie $\mathcal{L}$ has non-logical symbols $\{+, \times, <, 0, 1\}$. Let

$$\mathrm{Th}\mathbb{N} = \{\sigma \mid \sigma \text{ is a sentence satisfied by } \langle \mathbb{N}; +, \times, <, 0, 1 \rangle\}.$$

Consider the following set Σ of wffs:

$$\mathrm{Th}\mathbb{N} \cup \{x > \underbrace{1 + \ldots + 1}_{n \text{ times}} \mid n \geq 1\}.$$

We claim that Σ is finitely satisfiable. To see this, suppose that $\Sigma_0 \subseteq \Sigma$ is any finite subset; say, $\Sigma_0 = T \cup \{x > \underbrace{1 + \ldots + 1}_{n_1}, \ldots, x > \underbrace{1 + \ldots + 1}_{n_t}\}$, where $T \subseteq \mathrm{Th}\mathbb{N}$. Let $m = \max\{n_1, \ldots, n_t\}$ and let $s \colon V \to \mathbb{N}$ with $s(x) = m + 1$. Then $\mathbb{N}$ satisfies Σ_0 with s. By the Compactness Theorem, there exists a structure $\mathcal{A}$ for $\mathcal{L}$ and a function $s \colon V \to A$ such that $\mathcal{A}$ satisfies Σ with s. Thus $\mathcal{A}$ is a "model of artihmetic" containing the "infinite natural number" $s(x) \in A$.

Discussion of the order relation in $\mathcal{A}$....

Now we return to the systematic development of first order logic.

Definition 19.3. Let $\mathcal{A}, \mathcal{B}$ be structures for the language $\mathcal{L}$. A function $f \colon A \to B$ is an *isomprphism* iff the following conditions are satisfied.

1. f is a bijection.

2. For each n-ary predicate symbol P and any n-tuple $a_1, \ldots, a_n \in A$,

$$\langle a_1, \ldots, a_n \rangle \in P^{\mathcal{A}} \text{ iff } \langle f(a_1), \ldots, f(a_n) \rangle \in P^{\mathcal{B}}.$$

3. For each constant symbol c, $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$.

4. For each n-ary function symbol h and n-tuple $a_1, \ldots, a_n \in A$,
$$f(h^{\mathcal{A}}(a_1, \ldots, a_n)) = h^{\mathcal{B}}(f(a_1), \ldots, f(a_n)).$$

We write $\mathcal{A} \cong \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Theorem 19.4. *Suppose that $\varphi\colon A \to B$ is an isomorphism. If σ is any sentence, then $\mathcal{A} \models \sigma$ iff $\mathcal{B} \models \sigma$.*

In order to prove the above theorem, we must prove the following more general statement.

Theorem 19.5. *Suppose that $\varphi\colon A \to B$ is an isomorphism and $s\colon V \to A$. Then for any wff α*
$$\mathcal{A} \models \alpha[s] \quad \textit{iff} \quad \mathcal{B} \models \alpha[\varphi \circ s].$$

We shall make use of the following result.

Lemma 19.6. *With the above hypotheses, for each term t,*
$$\varphi(\bar{s}(t)) = (\overline{\varphi \circ s})(t).$$

Proof. Exercise. □

Proof of Theorem 19.5. We argue by induction of the complexity of α. First suppose that α is atomic, say $Pt_1 \ldots t_n$. Then
$$\begin{aligned} \mathcal{A} \models Pt_1 \ldots t_n[s] \quad & \text{iff} \quad \langle \bar{s}(t_1), \ldots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}} \\ & \text{iff} \quad \langle \varphi(\bar{s}(t_1)), \ldots, \varphi(\bar{s}(t_n)) \rangle \in P^{\mathcal{B}} \\ & \text{iff} \quad \langle (\overline{\varphi \circ s})(t_1)), \ldots, (\overline{\varphi \circ s})(t_n) \rangle \in P^{\mathcal{B}} \\ & \text{iff} \quad \mathcal{B} \models Pt_1 \ldots t_n[\varphi \circ s] \end{aligned}$$

Next suppose that α is $\neg\beta$. Then
$$\begin{aligned} \mathcal{A} \models \neg\beta[s] \quad & \text{iff} \quad \mathcal{A} \not\models \beta[s] \\ & \text{iff} \quad \mathcal{B} \not\models \beta[\varphi \circ s] \\ & \text{iff} \quad \mathcal{B} \models \neg\beta[\varphi \circ s] \end{aligned}$$

A similar argument deals with the case when α is $(\beta \implies \gamma)$.

Finally suppose that α is $\forall v\beta$. Then
$$\begin{aligned} \mathcal{A} \models \forall v\beta[s] \quad & \text{iff} \quad \mathcal{A} \models \beta[s(v|a)], \text{ for all } a \in A \\ & \text{iff} \quad \mathcal{B} \models \beta[\varphi \circ s(v|a)], \text{ for all } a \in A \\ & \text{iff} \quad \mathcal{B} \models \beta[(\varphi \circ s)(v|\varphi(a))], \text{ for all } a \in A \\ & \text{iff} \quad \mathcal{B} \models \beta[(\varphi \circ s)(v|b)], \text{ for all } b \in B \\ & \text{iff} \quad \mathcal{B} \models \forall v\beta[\varphi \circ s] \end{aligned}$$

□

Example 19.7. $\langle \mathbb{N}, < \rangle \ncong \langle \mathbb{Z}, < \rangle$.

Proof. Consider the sentence σ given by

$$(\exists x)(\forall y)(y = x \vee x < y).$$

Then $\langle \mathbb{N}, < \rangle \models \sigma$ and $\langle \mathbb{Z}, < \rangle \not\models \sigma$. Thus $\langle \mathbb{N}, < \rangle \ncong \langle \mathbb{Z}, < \rangle$. □

Example 19.8. $\langle \mathbb{Z}, < \rangle \ncong \langle \mathbb{Q}, < \rangle$.

Proof. Consider the sentence σ given by

$$(\forall x)(\forall y)(x < y \rightarrow (\exists z)(x < z \wedge z < y)).$$

□

Definition 19.9. Let T be a set of sentences.

1. $\mathcal{A}$ is a *model* for T iff $\mathcal{A} \models \sigma$ for every $\sigma \in T$.
2. $\mathrm{Mod}(T)$ is the class of all models of T.

Abbreviation If E is a binary predicate symbol, then we usually write xEy instead of Exy.

Example 19.10. Let T be the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \rightarrow yEx).$$

Then $\mathrm{Mod}(T)$ is the class of graphs.

Example 19.11. Let T be the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(\forall z)((xEy \wedge yEz) \rightarrow xEz)$$

$$(\forall x)(\forall y)(x = y \vee xEy \vee yEx)$$

Then $\mathrm{Mod}(T)$ is the class of linear orders.

Definition 19.12. A class $\mathcal{C}$ of structures is *axiomatizable* iff there is a set T of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$. If there exists a finite set T of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$, then $\mathcal{C}$ is *finitely axiomatizable*.

Example 19.13. The class of graphs is finitely axiomatizable.

Example 19.14. The class of infinite graphs is axiomatizable.

Proof. For each $n \geq 1$ let $\mathcal{O}_n$ be the sentence

"There exist at least n elements."

For example $\mathcal{O}_3$ is the sentence

$$(\exists x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge z \neq x).$$

Then $\mathcal{C} = \mathrm{Mod}(T)$, where T is the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \rightarrow yEx)$$

$$\mathcal{O}_n, \quad n \geq 1.$$

□

Question 19.15. Is the class of infinite graphs finitely axiomatizable?

Question 19.16. Is the class of finite graphs axiomatizable?

Another application of the Compactness Theorem...

Theorem 19.17. *Let T be a set of sentences in a first order language $\mathcal{L}$. If T has arbitrarily large finite models, then T has an infinite model.*

Proof. For each $n \geq 1$, let $\mathcal{O}_n$ be the sentence which says:

"There exist at least n elements."

Let Σ be the set of sentences $T \cup \{\mathcal{O}_n \mid n \geq 1\}$. We claim that Σ is finitely satisfiable. Suppose $\Sigma_0 \subseteq \Sigma$ is any finite subset. Then wlog

$$\Sigma_0 = T \cup \{\mathcal{O}_{n_1}, \ldots, \mathcal{O}_{n_t}\}.$$

Let $m = \max\{n_1, \ldots, n_t\}$. Then there exists a finite model $\mathcal{A}_0$ of T such that $\mathcal{A}_0$ has at least m elements. Clearly $\mathcal{A}_0$ satisfies Σ_0. By the Compactness Theorem, there exists a model $\mathcal{A}$ of Σ. Clearly $\mathcal{A}$ is an infinite model of T. □

Corollary 19.18. *The class $\mathcal{F}$ of finite graphs is* not *axiomatizable.*

Proof. Suppose T is a set of sentences such that $\mathcal{F} = \mathrm{Mod}(T)$. Clearly there are arbitrarly large finite graphs and hence T has arbitrarly large finite models. But this means that T has an infinite model, which is a contradiction. □

Corollary 19.19. *The class $\mathcal{C}$ of infinite graphs is not finitely axiomatizable.*

Proof. Suppose that there exists a finite set $T = \{\varphi_1, \dots, \varphi_n\}$ of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$. Consider the following set T' of sentences.

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \to yEx)$$

$$\neg(\varphi_1 \wedge \dots \wedge \varphi_n).$$

Then clearly $Mod(T')$ is the class of finite graphs, which is a contradiction. □

20 Valid sentences

Definition 20.1. Let Σ be a set of wffs and let φ be a wff. Then Σ *logically implies/semantically implies* φ iff for every structure $\mathcal{A}$ and for every function $s\colon V \to A$, if $\mathcal{A}$ satisfies Σ with s, then $\mathcal{A}$ satisfies φ with s. In this case we write $\Sigma \models \varphi$.

Definition 20.2. The wff φ is *valid* iff $\emptyset \models \varphi$; *i.e.*, for all structures $\mathcal{A}$ and functions $s\colon V \to A$, $\mathcal{A} \models \varphi[s]$.

Example 20.3. $\{\forall x P x\} \models Pc$.

Question 20.4. Suppose that Σ is an infinite set of wffs and that $\Sigma \models \varphi$. Does there exist a finite set $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$?

Answer Yes. We shall show that $\Sigma \models \varphi$ iff there exists a proof of φ from Σ. Such a proof will only use a finite subset $\Sigma_0 \subseteq \Sigma$.

We now return to the syntactic aspect of first order languages. We will next define rigorously the notion of a *deduction* or proof.

Notation Λ will denote the set of *logical axioms*. These will be defined explicitly a little later.

$$\text{eg} \quad (\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)).$$

Each logical axiom will be valid.

Definition 20.5. Let Γ be a set of wffs and φ a wff. A *deduction* of φ from Γ is a finite sequence of wffs

$$\langle \alpha_1, \dots, \alpha_n \rangle$$

such that $\alpha_n = \varphi$ and for each $1 \le i \le n$, either:

(a) $\alpha_i \in \Lambda \cup \Gamma$; or

(b) there exist $j, k < i$ such that α_k is $(\alpha_j \to \alpha_i)$.

Remark 20.6. In case (b), we have

$$\langle \alpha_1, \ldots, \alpha_j, \ldots, (\alpha_j \to \alpha_i), \ldots, \alpha_i, \ldots, \alpha_n \rangle$$

We say that α_i follows from α_j and $(\alpha_j \to \alpha_i)$ by modus ponens (MP).

Definition 20.7. φ is a *theorem* of Γ, written $\Gamma \vdash \varphi$, iff there exists a deduction of φ from Γ.

The two main results of this course...

Theorem 20.8 (Soundness). *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$*.*

Theorem 20.9 (Completeness (Godel)). *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*