21 The Logical Axioms Λ

φ is a *generalization* of ψ iff for some $n \geq 0$ and variables $x_1, \ldots, x_n$, we have that φ is

$$\forall x_1 \ldots \forall x_n \psi.$$

The logical axioms are all generalizations of all wffs of the following forms:

1. Tautologies.
2. $(\forall x\alpha \rightarrow \alpha_t^x)$, where t is a term which is substitutable for x in α.
3. $(\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta))$.
4. $(\alpha \rightarrow \forall x\alpha)$, where x doesn't appear free in α.
5. $x = x$.
6. $(x = y \rightarrow (\alpha \rightarrow \alpha'))$, where α is atomic and α' is obtained from α be replacing some (possibly none) of the occurrences of x by y.

Explanation 1. A tautology is a wff that can be obtained from a propostional tautology by substituting wffs for sentence symbols.

$$e.g. \quad (P \rightarrow \neg Q) \rightarrow (Q \rightarrow \neg P)$$

is a propositional tautology.

$$(\forall x\alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \neg \forall x\alpha)$$

is a first order tautology.

Explanation 2. α_t^x is the result of replacing each free occurrence of x by t. We say that t is *substitutable* for x in α iff no variable of t gets bound by a quantifier in α_t^x.

e.g. Let α be $\neg \forall y(x = y)$. Then y is *not* substitutable for x in α. Note that in this case

$$\forall x\alpha \rightarrow \alpha_t^x$$

becomes

$$\forall x \neg \forall y(x = y) \rightarrow \neg \forall y(y = y)$$

which is *not* valid. So we need the above restriction.

Explanation 4. A typical example is

$$Pyz \rightarrow \forall x Pyz.$$

Here "$\forall x$" is a "dummy quantifier" which does nothing. Note that

$$x = 0 \rightarrow \forall x(x = 0)$$

is *not* valid. So we need the above restriction.

22 Some examples of deductions

Example 22.1. $\vdash (Px \to \exists y Py)$

Proof. Note that $(Px \to \exists y Py)$ is an abbreviation of $(Px \to \neg\forall y \neg Py)$. The following is a deduction from $\emptyset$.

1. $(\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py)$ [Axiom 1]
2. $(\forall y \neg Py \to \neg Px)$ [Axiom 2]
3. $(Px \to \neg \forall y \neg Py)$ [MP, 1, 2]

□

Example 22.2. $\vdash \forall x(Px \to \neg\forall y \neg Py)$

Proof. The following is a deduction from $\emptyset$.

1. $\forall x((\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py))$ [Axiom 1]
2. $\forall x(\forall y \neg Py \to \neg Px)$ [Axiom 2]
3. $\forall x((\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py)) \to (\forall x(\forall y \neg Py \to \neg Px) \to \forall x(Px \to \neg \forall y \neg Py))$ [Axiom 3]
4. $\forall x(\forall y \neg Py \to \neg Px) \to \forall x(Px \to \neg \forall y \neg Py)$ [MP, 1, 3]
5. $\forall x(Px \to \neg \forall y \neg Py))$ [MP, 2, 4]

□

23 Soundness Theorem

Theorem 23.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

We shall make use of the following result.

Lemma 23.2. *Every logical axiom $\varphi \in \Lambda$ is valid*

Proof. We just consider the case where φ has the form

$$(\alpha \to \forall x \alpha)$$

where x isn't free in α. Let $\mathcal{A}$ be any structure and $s \colon V \to A$. If $\mathcal{A} \not\models \alpha[s]$, then $\mathcal{A} \models (\alpha \to \forall x \alpha)[s]$. So suppose that $\mathcal{A} \models \alpha[s]$. Let $a \in A$ be any element. Then s and $s(x|a)$ agree on the free variables of α. Hence $\mathcal{A} \models \alpha[s(x|a)]$ and so $\mathcal{A} \models (\alpha \to \forall x \alpha)[s]$. □

Remark 23.3. The other cases are equally easy, except for the case of

$$(\forall x\alpha \rightarrow \alpha^x_t)$$

which is harder. We will give a detailed proof of this case later.

Exercise 23.4. Show that

$$(\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \beta))$$

is valid.

Proof of the Soundness Theorem. We argue by induction on the minimal length $n \geq 1$ of a deduction that if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

First suppose that $n = 1$. Then $\varphi \in \Gamma \cup \Lambda$. If $\varphi \in \Gamma$ then clearly $\Gamma \models \varphi$. If $\varphi \in \Lambda$, then the lemma (23.2) says that φ is valid. Thus $\emptyset \models \varphi$ and so $\Gamma \models \varphi$.

Now suppose that $n > 1$. Let

$$\langle \alpha_1, \dots, \alpha_n = \varphi \rangle$$

be a deduction of φ from Γ. Then φ must follow from MP from two earlier wffs θ and $(\theta \rightarrow \varphi)$. Note that proper initial segments of deductions from Γ are also deductions from Γ. Thus $\Gamma \vdash \theta$ and $\Gamma \vdash (\theta \rightarrow \varphi)$ via deductions of length less than n. By induction hypothesis, $\Gamma \models \theta$ and $\Gamma \models (\theta \rightarrow \varphi)$. Let $\mathcal{A}$ be any structure and $s\colon V \rightarrow A$. Suppose that $\mathcal{A}$ satisfies Γ with s. Then $\mathcal{A} \models \theta[s]$ and $\mathcal{A} \models (\theta \rightarrow \varphi)[s]$. Hence $\mathcal{A} \models \varphi[s]$. Thus $\Gamma \models \varphi$. □

Definition 23.5. A set Γ of wffs is *inconsistent* iff there exists a wff β such that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Otherwise, Γ is *consistent*.

Corollary 23.6. *If Γ is satisfiable, then Γ is consistent.*

Proof. Suppose that Γ is satisfiable. Let $\mathcal{A}$ satisfy Γ with $s\colon V \rightarrow A$. Now suppose that Γ is inconsistent; say $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. By Soundness $\Gamma \models \beta$ and $\Gamma \models \neg\beta$. But this means that $\mathcal{A} \models \beta[s]$ and $\mathcal{A} \models \neg\beta[s]$. which is a contradiction. □