24 Meta-theorems

Now we turn to the proof of the Completeness Theorem. First we need to prove a number of "Meta-Theorems".

Theorem 24.1 (Generalization). *If $\Gamma \vdash \varphi$ and x doesn't occur free in any wff of Γ, then $\Gamma \vdash \forall x\varphi$.*

Remark 24.2. Note that if c is a constant symbol, then

$$\{x = c\} \vdash x = c.$$

However,

$$\{x = c\} \nvdash \forall x(x = c).$$

How do we know this? By the Soundness Theorem, it is enough to show that

$$\{x = c\} \not\models \forall x(x = c).$$

Proof of Generalization Theorem. We argue by induction on the minimal length n of a deduction of φ from Γ that $\Gamma \vdash \forall x\varphi$.

First suppose that $n = 1$. Then $\varphi \in \Gamma \cup \Lambda$.

Case 1 Suppose that $\varphi \in \Lambda$. Then $\forall x\varphi \in \Lambda$ and so $\Gamma \vdash \forall x\varphi$.

Case 2 Suppose that $\varphi \in \Gamma$. Then x doesn't occur free in φ and so $(\varphi \to \forall x\varphi) \in \Lambda$. Hence the following is a deduction of $\forall x\varphi$ from Γ.

1. φ [in Γ]
2. $\varphi \to \forall x\varphi$ [Ax 4]
3. $\forall x\varphi$ [MP, 1, 2]

Now suppose that $n > 1$. Then is a deduction of minimal length, φ follows from earlier wffs θ and $(\theta \to \varphi)$ by MP. By induction hypothesis, $\Gamma \vdash \forall x\theta$ and $\Gamma \vdash \forall x(\theta \to \varphi)$. Hence the following is a deduction of $\forall x\varphi$ from Γ.

1. … deduction of $\forall x\theta$ from Γ.

n. $\forall x\theta$

n+1. … deduction of $\forall x(\theta \to \varphi)$ from Γ.

n+m. $\forall x(\theta \to \varphi)$

n+m+1. $\forall x(\theta \to \varphi) \to (\forall x\theta \to \forall x\varphi)$ [Ax 3] n+m+2. $\forall x\theta \rightarrow \forall x\varphi$ [MP, $n+m$, $n+m+1$]

n+m+3. $\forall x\varphi$ [MP, n, $n+m+2$]

□

Definition 24.3. $\{\alpha_1, \ldots, \alpha_n\}$ *tautologically implies* β iff

$$(\alpha_1 \rightarrow (\alpha_2 \rightarrow \ldots (\alpha_n \rightarrow \beta) \ldots))$$

is a tautology.

Theorem 24.4 (Rule T). *If* $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ *and* $\{\alpha_1, \ldots, \alpha_n\}$ *tautologically implies* β*, the* $\Gamma \vdash \beta$*.*

Proof. Obvious, via repeated applications of MP. □

Theorem 24.5 (Deduction). *If* $\Gamma \cup \{\gamma\} \vdash \varphi$*, then* $\Gamma \vdash (\gamma \rightarrow \varphi)$*.*

Proof. We argue by induction on the minimal length n of a deduction of φ from $\Gamma \cup \{\gamma\}$.

First suppose that $n = 1$.

Case 1 Suppose that $\varphi \in \Gamma \cup \Lambda$. Then the following is a deduction from Γ.

1. φ [in $\Gamma \cup \Lambda$]
2. $(\varphi \rightarrow (\gamma \rightarrow \varphi))$ [Ax 1]
3. $(\gamma \rightarrow \varphi)$ [MP, 1, 2]

Case 2 Suppose that $\varphi = \gamma$. In this case $(\gamma \rightarrow \varphi)$ is a tautology and so $\Gamma \vdash (\gamma \rightarrow \varphi)$.

Now suppose that $n > 1$. Then in a deduction of minimal length φ follows from earlier wffs θ and $(\theta \rightarrow \varphi)$ by MP. By induction hypothesis, $\Gamma \vdash (\gamma \rightarrow \theta)$ and $\Gamma \vdash (\gamma \rightarrow (\theta \rightarrow \varphi))$. Clearly $\{(\gamma \rightarrow \theta), (\gamma \rightarrow (\theta \rightarrow \varphi))\}$ tautologically implies $(\gamma \rightarrow \varphi)$. By Rule T, $\Gamma \vdash (\gamma \rightarrow \varphi)$. □

Theorem 24.6 (Contraposition). $\Gamma \cup \{\varphi\} \vdash \neg\psi$ *iff* $\Gamma \cup \{\psi\} \vdash \neg\varphi$*.*

Proof. Suppose that $\Gamma \cup \{\varphi\} \vdash \neg\psi$. By the deduction theorem $\Gamma \vdash (\varphi \rightarrow \neg\psi)$. By Rule T, $\Gamma \vdash (\psi \rightarrow \neg\phi)$. Hence $\Gamma \cup \{\psi\} \vdash \psi$ amd $\Gamma \cup \{\psi\} \vdash (\psi \rightarrow \neg\varphi)$. By Rule T, $\Gamma \cup \{\gamma\} \vdash \neg\varphi$.

The other direction is similar. □

Theorem 24.7 (Reductio Ad Absurdum). *If* $\Gamma \cup \{\varphi\}$ *is inconsistent, then* $\Gamma \vdash \neg\varphi$*.*

Proof. Suppose that $\Gamma \cup \{\varphi\} \vdash \beta$ and $\Gamma \cup \{\varphi\} \vdash \neg\beta$. By the Deduction Theorem , $\Gamma \vdash (\varphi \rightarrow \beta)$ and $\Gamma \vdash (\varphi \rightarrow \neg\beta)$. Since $\{(\varphi \rightarrow \beta), (\varphi \rightarrow \neg\beta)\}$ tautologically implies $\neg\varphi$, Rule T gives $\Gamma \vdash \neg\varphi$. □

Remark 24.8. If Γ is inconsistent, then $\Gamma \vdash \alpha$ for ever wff α.

Proof. Suppose that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Clearly

$$(\beta \rightarrow (\neg\beta \rightarrow \alpha))$$

is a tautology. By Rule T, $\Gamma \vdash \alpha$. □

25 Applications: some theorems about equality

Eq 1.
$\vdash \forall x(x = x)$

Proof. This is a logical axiom. □

Eq 2.
$\vdash \forall x \forall y(x = y \rightarrow y = x)$

Proof.

1. $\vdash x = y \rightarrow (x = x \rightarrow y = x)$ [Ax 6]
2. $\vdash x = x$ [Ax 5]
3. $\vdash x = y \rightarrow y = x$ [Rule T, 1, 2]
4. $\vdash \forall y(x = y \rightarrow y = x)$ [Gen, 3]
5. $\vdash \forall x \forall y(x = y \rightarrow y = x)$ [Gen, 4]

□

Eq 3.
$\vdash \forall x \forall y \forall z(x = y \rightarrow (y = z \rightarrow x = z))$

Proof.

1. $\vdash y = x \rightarrow (y = z \rightarrow x = z)$ [Ax 6]
2. $\vdash x = y \rightarrow y = x$ [Shown in proof of Eq 2]
3. $\vdash x = y \rightarrow (y = z \rightarrow x = z)$ [Rule T, 1, 2]
4. $\vdash \forall x \forall y \forall z(x = y \rightarrow (y = z \rightarrow x = z))$ [Gen cubed, 3]

□

26 Generalization on constants

Theorem 26.1 (Generalization on constants). *Assume that $\Gamma \vdash \varphi$ and that c is a constant symbol which doesn't occur in Γ. Then there exists a variable y (which doesn't occur in φ) such that $\Gamma \vdash \forall y \varphi^c_y$.*

Furthermore, there exists a deduction of $\forall y \varphi^c_y$ from Γ in which c doesn't occur.

Remark 26.2. Intuitively, suppose that Γ says nothing about c and that $\Gamma \vdash \varphi(c)$. Then $\Gamma \vdash \forall y \varphi(y)$. In other words, to prove $\forall y \varphi(y)$, let c be arbitrary and prove $\varphi(c)$.

Remark 26.3. Suppose that Γ is a consistent set of wffs in the language $\mathcal{L}$. Let $\mathcal{L}^+$ be the language obtained by adding a new constant symbol c. Then Γ is still consistent in $\mathcal{L}^+$.

Why? Suppose not. Then there exists a wff β in $\mathcal{L}^+$ such that $\Gamma \vdash \beta \wedge \neg \beta$ in $\mathcal{L}^+$. By the above theorem, for some variable y which doesn't occur in β,

$$\Gamma \vdash \forall y (\beta^c_y \wedge \neg \beta^c_y)$$

via a deduction that doesn't involve c. Since

$$\forall y (\beta^c_y \wedge \neg \beta^c_y) \rightarrow (\beta^c_y \wedge \neg \beta^c_y)$$

is a logical axiom,

$$\Gamma \vdash \beta^c_y \wedge \neg \beta^c_y$$

in $\mathcal{L}$. This implies that Γ is inconsistent in $\mathcal{L}$, which is a contradiction.

Proof of Generalization on Constants. Suppose that

$$(*) \quad \langle \alpha_1, \dots, \alpha_n \rangle$$

is a deduction of φ from Γ. Let y be a variable which doesn't occur in any of the α_i. We claim that

$$(**) \quad \langle (\alpha_1)^c_y, \dots, (\alpha_n \rangle)^c_y$$

is a deduction of φ^c_y from Γ. We shall prove that, for all $i \leq n$, either $(\alpha_i)^c_y \in \Gamma \cup \Lambda$ or $(\alpha_i)^c_y$ follows from earlier wffs in $(**)$ via MP.

Case 1 Suppose that $\alpha_i \in \Gamma$. Since c doesn't occur in Γ, it follows that $(\alpha_i)^c_y = \alpha_i \in \Gamma$.

Case 2 Suppose that $\alpha_i \in \Lambda$. Then it is easily checked that $(\alpha_i)^c_y \in \Lambda$.

Case 3 Suppose there exist $j, k < i$ such that α_k is $(\alpha_j \rightarrow \alpha_i)$. Then $(\alpha_k)^c_y$ is $((\alpha_j)^c_y \rightarrow (\alpha_i)^c_y)$. Hence $(\alpha_i)^c_y$ follows from $(\alpha_k)^c_y$ and $(\alpha_j)^c_y$ by MP.

Let Φ be the finite subset of Γ which occurs in $(**)$. Then $\Phi \vdash \varphi^c_y$ via a deduction in which c doesn't occur. By the Generalization Theorem, since y doesn't occur free in Φ, it follows that $\Phi \vdash \forall y \varphi^c_y$ via a deduction in which c doesn't occur. It follows that $\Gamma \vdash \forall y \varphi^c_y$ via a deduction in which c doesn't occur. □

Exercise 26.4. 1. Show by induction on φ that if y doesn't occur in φ, then x is substitutable for y in φ^x_y and $(\varphi^x_y)^y_x = \varphi$.

2. Find a wff φ such that $(\varphi^x_y)^y_x \neq \varphi$.

Corollary 26.5. *Suppose that $\Gamma \vdash \varphi^x_c$, where c is a constant symbol that doesn't occur in Γ or φ. Then $\Gamma \vdash \forall x\varphi$, via a deduction in which c doesn't occur.*

Proof. By the above theorem, $\Gamma \vdash \forall y(\varphi^x_c)^c_y$ for some variable y which doesn't occur in φ^x_c. Since c doesn't occur in φ, $(\varphi^x_c)^c_y = \varphi^x_y$. Thus $\Gamma \vdash \forall y\varphi^x_y$. By the exercise, the following is a logical axiom: $\forall y\varphi^x_y \rightarrow \varphi$. Thus $\forall y\varphi^x_y \vdash \phi$. Since x doesn't occur free in $\forall y\varphi^x_y$, Generalization gives that $\forall y\varphi^x_y \vdash \forall x\varphi$. Hence Deduction yields that $\vdash \forall y\varphi^x_y \rightarrow \forall x\varphi$. Since $\Gamma \vdash \forall y\varphi^x_y$, Rule T gives $\Gamma \vdash \forall x\phi$. $\square$

Theorem 26.6 (Existence of Alphabetic Variants). *Let φ be a wff, t a term and x a variable. Then there exists a wff φ' (which differs from φ only in the choice of quantified variables) such that:*

(a) $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$.

(b) t is substitutable for x in φ'.

Proof Omitted

27 Completeness

Now we are ready to begin the proof of:

Theorem 27.1 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

We shall base our strategy on the following observation.

Proposition 27.2. *The following statements are equivalent:*

(a) The Completeness Theorem: i.e. if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

(b) If Γ is a consistent set of wffs, then Γ is satisfiable.

Proof. (a) $\Rightarrow$ (b)

Suppose that Γ is consistent. Then there exists a wff φ such that $\Gamma \nvdash \varphi$. By Completeness, $\Gamma \not\models \varphi$. Hence there exists a structure $\mathcal{A}$ and a function $s\colon V \rightarrow A$ such that $\mathcal{A}$ satisfies Γ with s and $\mathcal{A} \not\models \varphi[s]$. In particular, Γ is satisfiable.

(b) $\Rightarrow$ (a)

Suppose that $\Gamma \nvdash \varphi$. Applying Reductio ad Absurdum, $\Gamma \cup \{\neg\varphi\}$ is consistent. It follows that $\Gamma \cup \{\neg\varphi\}$ is satisfiable and hence $\Gamma \not\models \varphi$. $\square$

Now we prove:

Theorem 27.3 (Completeness'). *If Γ is a consistent set of wffs in a countable language $\mathcal{L}$, then there exists a countable structure $\mathcal{A}$ and $s\colon V \to A$ such that $\mathcal{A}$ satisfies Γ with s.*

Proof. **Step 1** Expand $\mathcal{L}$ to a larger language $\mathcal{L}^+$ by adding a countably infinte set of new constant symbols. Then Γ remains consistent as a set of wffs in $\mathcal{L}^+$.

Proof of Step 1. Suppose not. Then there exists a wff β of $\mathcal{L}^+$ such that $\Gamma \vdash \beta \wedge \neg\beta$ in $\mathcal{L}^+$. Suppose that $c_1, \ldots, c_n$ includes the new constants (if any) which appear in β. By Generalization on Constants, there are variables $y_1, \ldots, y_n$ such that:

(a) $\Gamma \vdash \forall y_1 \ldots \forall y_n (\beta' \wedge \neg\beta')$, where β' is the result of replacing each c_i by y_i; and

(b) the deduction doesn't involve any new constants.

Since y_i is substitutable for y_i in β', we obtain that $\Gamma \vdash \beta' \wedge \neg\beta'$. But this means that Γ is inconsistent in the original language $\mathcal{L}$, which is a contradiction. □

Step 2 (We add witnesses to existential wffs.) Let

$$\langle \varphi_1, x_1 \rangle, \langle \varphi_2, x_2 \rangle, \ldots, \langle \varphi_n, x_n \rangle, \ldots$$

enumerate all pairs $\langle \varphi, x \rangle$, where φ is a wff of $\mathcal{L}^+$ and x is a variable. Let θ_1 be the wff

$$\neg\forall x_1 \varphi_1 \to (\neg\varphi_1)^{x_1}_{c_1},$$

where c_1 is the first new constant which doesn't occur in φ_1. If $n > 1$, then θ_n is the wff

$$\neg\forall x_n \varphi_n \to (\neg\varphi_n)^{x_n}_{c_n},$$

where c_n is the first new constant which doesn't occur in $\{\varphi_1, \ldots, \varphi_n\} \cup \{\theta_1, \ldots, \theta_{n-1}\}$. Let

$$\Theta = \Gamma \cup \{\theta_n \mid n \geq 1\}.$$

Claim 27.4. Θ is consistent.

Proof. Suppose not. Let $n \geq 0$ be the least integer such that $\Gamma \cup \{\theta_1, \ldots, \theta_{n+1}\}$ is inconsistent. By Reductio ad Absurdum,

$$\Gamma \cup \{\theta_1, \ldots, \theta_n\} \vdash \neg\theta_{n+1}.$$

Recall that θ_{n+1} has the form

$$\neg\forall x\varphi\rightarrow\neg\varphi_c^x.$$

By Rule T,

$$\Gamma \cup \{\theta_1, \dots, \theta_n\} \vdash \neg\forall x\varphi.$$

and

$$\Gamma \cup \{\theta_1, \dots, \theta_n\} \vdash \varphi_c^x.$$

Since c doesn't occur in $\Gamma \cup \{\theta_1, \dots, \theta_n\} \cup \{\varphi\}$, we have that

$$\Gamma \cup \{\theta_1, \dots, \theta_n\} \vdash \forall x\varphi.$$

But this contradicts the minimality of n, or the consistency of Γ if $n = 0$. □

Step 3 We extend Θ to a consistent set of wffs Δ such that for every wff φ of $\mathcal{L}^+$, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

Proof. Let $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ enumerate all the wffs of $\mathcal{L}^+$. We define inductively an increasing sequence of consistent sets of wffs

$$\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_n \subseteq \dots$$

as follows

- $\Delta_0 = \Theta$
- Suppose that Δ_n has been defined. If $\Delta_n \cup \{\alpha_{n+1}\}$ is consistent, then we set $\Delta_{n+1} = \Delta \cup \{\alpha_{n+1}\}$.

  Otherwise, if $\Delta_n \cup \{\alpha_{n+1}\}$ is inconsistent, then $\Delta \vdash \neg\alpha_{n+1}$ so we can set $\Delta_{n+1} = \Delta \cup \{\neg\alpha_{n+1}\}$.

Finally let $\Delta = \cup_{n\geq 0}\Delta_n$. Clearly Δ satisfies our requirements. □

Notice that Δ is deductively closed; *i.e.* if $\Delta \vdash \varphi$, then $\varphi \in \Delta$. Otherwise, $\neg\varphi \in \Delta$ and so $\Delta \vdash \varphi$ and $\Delta \vdash \neg\varphi$, which contradicts the consistency of Δ.

Step 4 For each of the following wffs φ, $\Delta \vdash \varphi$ and so $\varphi \in \Delta$.

Eq 1 $\forall x(x = x)$.

Eq 2 $\forall x\forall y(x = y\rightarrow y = x)$.

Eq 3 $\forall x\forall y\forall z((x = y\wedge y = z)\rightarrow x = z)$.

Eq 4 For each n-ary predicate symbol P

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow (Px_1 \dots x_n \leftrightarrow Py_1 \dots y_n)$$

Eq 5 For each n-ary function symbol f

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow (fx_1 \dots x_n = fy_1 \dots y_n)$$

Similarly, since Δ is deductively closed and $\forall x \forall y (x = y \rightarrow y = x) \in \Delta$, if t_1, t_2 are any terms, then $(t_1 = t_2 \rightarrow t_2 = t_1) \in \Delta$ *etc.*.

Step 5 We construct a structure $\mathcal{A}$ for $\mathcal{L}^+$ as follows.

Let T be the set of terms in $\mathcal{L}^+$. Define a relation E on T by

$$t_1 E t_2 \quad \text{iff} \quad (t_1 = t_2) \in \Delta.$$

Claim 27.5. *E is an equivalence relation.*

Proof. Suppose that $t \in T$. Then $(t = t) \in \Delta$ and so tEt. Thus E is reflexive.

Next suppose that $t_1 E t_2$. Then $(t_1 = t_2) \in \Delta$. Since $(t_1 = t_2 \rightarrow t_2 = t_1) \in \Delta$, it follows that $(t_2 = t_1) \in \Delta$. Thus $t_2 E t_1$ and so E is symmetric.

Similarly E is transitive. □

Definition 27.6. For each $t \in T$, let

$$[t] = \{s \in T \mid tEs\}.$$

Then we define

$$A = \{[t] \mid t \in T\}.$$

Definition 27.7. For each n-ary predicate symbol P, we define an n-ary relation $P^{\mathcal{A}}$ on A by

$$\langle [t_1], \dots, [t_n] \rangle \in P^{\mathcal{A}} \;\text{ iff }\; Pt_1 \dots t_n \in \Delta.$$

Claim 27.8. *$P^{\mathcal{A}}$ is well-defined.*

Proof. Suppose that $[s_1] = [t_1], \dots, [s_n] = [t_n]$. We must show that

$$Ps_1 \dots s_n \in \Delta \;\text{ iff }\; Pt_1 \dots t_n \in \Delta.$$

By assumption, $(s_1 = t_1) \in \Delta, \dots, (s_n = t_n) \in \Delta$. Since

$$[(s_1 = t_1 \wedge \dots \wedge s_n = t_n) \rightarrow (Ps_1 \dots s_n \leftrightarrow Pt_1 \dots t_n)] \in \Delta,$$

the result follows. □

Definition 27.9. For each constant symbol c, $c^{\mathcal{A}} = [c]$.

Definition 27.10. For each n-ary function symbol f, we define an n-ary operation $f^{\mathcal{A}} \colon A^n \to A$ by

$$f^{\mathcal{A}}([t_1], \dots, [t_n]) = [ft_1 \dots t_n].$$

Claim 27.11. $f^{\mathcal{A}}$ is well-defined.

Proof. Similar. □

Finally we define $s \colon V \to A$ by $s(x) = [x]$.

Claim 27.12 (Target). For every wff φ of $\mathcal{L}^+$,

$$\mathcal{A} \models \varphi[s] \text{ iff } \varphi \in \Delta.$$

We shall make use of the following result.

Claim 27.13. For each term $t \in T$, $\bar{s}(t) = [t]$.

Proof. By definition, the result holds when t is a variable or a constant symbol. Suppose that t is $ft_1 \dots t_n$. Then by induction hypothesis, $\bar{s}(t_1) = [t_1], \dots, \bar{s}(t_n) = [t_n]$. Hence

$$\begin{aligned} \bar{s}(ft_1 \dots t_n) &= f^{\mathcal{A}}(\bar{s}(t_1), \dots, \bar{s}(t_1)) \\ &= f^{\mathcal{A}}([t_1], \dots, [t_1]) \\ &= [ft_1, \dots, t_1] \end{aligned}$$

□

Proof of Target Claim. We argue by induction on the complexity of φ. First suppose that φ is atomic.

Case 1 Suppose that φ is $t_1 = t_2$. Then

$$\begin{aligned} \mathcal{A} \models (t_1 = t_2)[s] \quad &\text{iff} \quad \bar{s}(t_1) = \bar{s}(t_2) \\ &\text{iff} \quad [t_1] = [t_2] \\ &\text{iff} \quad (t_1 = t_2) \in \Delta \end{aligned}$$

Case 2 Suppose that φ is $Pt_1 \dots t_n$. Then

$$\begin{aligned} \mathcal{A} \models Pt_1 \dots t_n[s] \quad &\text{iff} \quad \langle \bar{s}(t_1), \dots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}} \\ &\text{iff} \quad \langle [t_1], \dots, [t_n] \rangle \in P^{\mathcal{A}} \\ &\text{iff} \quad Pt_1 \dots t_n \in \Delta \end{aligned}$$

Next we consider the case when φ isn't atomic.

Case 3 Suppose that φ is $\neg\psi$. Then

$$\begin{aligned} \mathcal{A} \models \neg\psi[s] \quad &\text{iff} \quad \mathcal{A} \not\models \psi[s] \\ &\text{iff} \quad \psi \notin \Delta \\ &\text{iff} \quad \neg\psi \in \Delta \end{aligned}$$

Case 4 The case where φ is $(\theta \rightarrow \psi)$ is similar.

Case 5 Finally suppose that φ is $\forall x\psi$. We shall make use of the following result.

Lemma 27.14 (Substitution). *If the term t is substitutable for x in ψ, then*

$$\mathcal{A} \models \psi^x_t[s] \quad \textit{iff} \quad \mathcal{A} \models \psi[s(x|\bar{s}(t))].$$

Proof. Omitted. □

Recall that φ is $\forall x\psi$. By construction, for some constant c,

$$(\neg\forall\psi \rightarrow \neg\psi^x_c) \in \Delta \quad (*)$$

First suppose that $\mathcal{A} \models \forall x\psi[s]$. Then, in particular, $\mathcal{A} \models \psi[s(x|[c])]$ and so $\mathcal{A} \models \psi[s(x|\bar{s}(c))]$. By the Substitution Lemma, $\mathcal{A} \models \psi^x_c[s]$. Hence by induction hypothesis, $\psi^x_c \in \Delta$ and so $\neg\psi^x_c \notin \Delta$. By (*), $\neg\forall x\psi \notin \Delta$ and so $\forall x\psi \in \Delta$.

Conversely, suppose that $\mathcal{A} \not\models \forall x\psi[s]$. Then there exists a term $t \in T$ such that $\mathcal{A} \not\models \psi[s(x|[t])]$. Thus $\mathcal{A} \not\models \psi[s(x|\bar{s}(t))]$. Let ψ' be an alphabetic variant of ψ such that t is substitutable for x in ψ'. Then $\mathcal{A} \not\models \psi'[s(x|\bar{s}(t))]$. By the Substitution Lemma, $\mathcal{A} \not\models (\psi')^x_t[s]$. By induction hypothesis, $(\psi')^x_t \notin \Delta$. Since t is substitutable for x in ψ' is follows that $(\forall x\psi' \rightarrow (\psi')^x_t) \in \Delta$. Hence $\forall x\psi' \notin \Delta$ and so $\forall x\psi \notin \Delta$. □

Finally let $\mathcal{A}_0$ be the structure for $\mathcal{L}$ obtained from $\mathcal{A}$ by forgetting the interpretations of the new constant symbols. Then $\mathcal{A}_0$ satisfies Γ with s.

This completes the proof of the Completeness Theorem. □

Corollary 27.15. $\Gamma \models \varphi$ *iff* $\Gamma \vdash \varphi$. □

Theorem 27.16. *Let Γ be a set of wffs in a countable first order language. If Γ is finitely satisfiable, then Γ is satisfiable in some countable structure.*

Proof. Suppose that every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable. By Soundness, every finite subset $\Gamma_0 \subseteq$ is consistent. Hence Γ is consistent. By Completeness, Γ is satisfiable in some countable structure. □

Theorem 27.17. *Let T be a set of sentences in a first order language $\mathcal{L}$. If the class $\mathcal{C} = \mathrm{Mod}(T)$ is finitely axiomatizable, then there exists a finite subset $T_0 \subseteq T$ such that $\mathcal{C} = \mathrm{Mod}(T_0)$.*

Proof. Suppose that $\mathcal{C} = \mathrm{Mod}(T)$ is finitely axiomatizable. Then there exists a sentence σ such that $\mathcal{C} = \mathrm{Mod}(\sigma)$. Since $\mathrm{Mod}(T) = \mathrm{Mod}(\sigma)$, it follows that $T \models \sigma$. By the Completeness Theorem, $T \vdash \sigma$ and hence there exists a finite subset $T_0 \subseteq T$ such that $T_0 \vdash \sigma$. By Soundness, $T_0 \models \sigma$. Hence

$$\mathcal{C} = \mathrm{Mod}(T) \subseteq \mathrm{Mod}(T_0) \subseteq \mathrm{Mod}(\sigma) = \mathcal{C}$$

and so $\mathcal{C} = \mathrm{Mod}(T_0)$. □

Definition 27.18. Let $\mathcal{A}$, $\mathcal{B}$ be structures for the first-order language $\mathcal{L}$. Then $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, written $\mathcal{A} \equiv \mathcal{B}$, iff for every sentence σ of $\mathcal{L}$,

$$\mathcal{A} \models \sigma \text{ iff } \mathcal{B} \models \sigma.$$

Remark 27.19. If $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$. Howevery, the converse does not hold, *e.g.* consider a nonstandard model of arithmetic.

Definition 27.20. A consistent set of sentences T is said to be *complete* iff for every sentence σ, either $T \vdash \sigma$ or $T \vdash \neg\sigma$.

Example 27.21. Let $\mathcal{A}$ be any structure and let

$$\mathrm{Th}(\mathcal{A}) = \{\sigma \mid \sigma \text{ is a sentence such that } \mathcal{A} \models \sigma\}.$$

Then $\mathrm{Th}(\mathcal{A})$ is a complete theory.

Theorem 27.22. *If T is a complete theory in the first-order language $\mathcal{L}$ and $\mathcal{A}$, $\mathcal{B}$ are models of T, then $\mathcal{A} \equiv \mathcal{B}$.*

Proof. Let σ be any sentence. Then either $T \vdash \sigma$ or $T \vdash \neg\sigma$. Suppose that $T \vdash \sigma$. By Soundness, $T \models \sigma$. Hence $\mathcal{A} \models \sigma$ and $\mathcal{B} \models \sigma$. Similarly if $T \vdash \neg\sigma$, then $\mathcal{A} \models \neg\sigma$ and $\mathcal{B} \models \neg\sigma$. □

Theorem 27.23 (Los-Vaught). *Let T be a consistent theory in a countable language $\mathcal{L}$. Suppose that*

(a) T has no finite models.

(b) If $\mathcal{A}$, $\mathcal{B}$ are countably infinite models of T, then $\mathcal{A} \cong \mathcal{B}$.

Then T is complete.

Proof. Suppose not. Then there exists a sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$. Hence $T \cup \{\neg\sigma\}$ and $T \cup \{\sigma\}$ are both consistent. By Completeness, there exists countable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \models T \cup \{\neg\sigma\}$ and $\mathcal{B} \models T \cup \{\sigma\}$. By (a), $\mathcal{A}$ and $\mathcal{B}$ must be countably infinite. Hence, by (b), $\mathcal{A} \cong \mathcal{B}$. But this contradicts the fact that $\mathcal{A} \models \neg\sigma$ and $\mathcal{B} \models \sigma$. □

Corollary 27.24. *Let T_{DLO} be the theory of dense linear orders without endpoints. Then T_{DLO} is complete.*

Proof. Clearly T_{DLO} has no finite models. Also, if $\mathcal{A}$, $\mathcal{B}$ are countable dense linear orders without endopints, then $\mathcal{A} \cong \mathcal{B}$. Hence T_{DLO} is complete. □

Corollary 27.25. $\langle \mathbb{Q}, < \rangle \equiv \langle \mathbb{R}, < \rangle$.

Proof. $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ are both models of the complete theory T_{DLO}. □

The rationals $\langle \mathbb{Q}, < \rangle$ are a countable linear order in which "every possible finite configuration is realized."