Feel free to work with other students on these problems. However all written work should be your own. Also, be sure to give written credit, on the assignment, for any ideas you get from other people.

All proofs should be as short and clear as possible. If you deviate from the style of proof given in the notes you should only do so consciously and for good reason.

Exercise 11.1. Show that a wff φ of the form

$$((\forall x\alpha \rightarrow \forall x\beta) \rightarrow \forall x(\alpha \rightarrow \beta))$$

need not be valid. (Find wffs α and β and a structure $\mathcal{A}$ so that φ is not satisfied by $\mathcal{A}$.)

Exercise 11.2.

1. Show by induction on φ that if y doesn't occur in φ, then x is substitutable for y in φ_y^x and $(\varphi_y^x)_x^y = \varphi$.
2. Find a wff φ such that $(\varphi_y^x)_x^y \neq \varphi$.

Exercise 11.3 (Optional). Let $\mathcal{L}$ be the first order language with only one non-logical symbol: the two-place predicate symbol S. Suppose that

$$\Gamma = \{\exists v_1 \forall v_2((\neg S v_2 v_2) \leftrightarrow S v_1 v_2)\}.$$

Show that Γ deduces a contradiction. (This is a version of the *barber paradox*.)

Exercise 11.4. Let $\mathcal{L}$ have the following non-logical symbols:

- a binary predicate symbol $<$; and
- a unary function symbol s.

Let T be the following set of sentences in the language $\mathcal{L}$:

1. $\forall x(\neg x < x)$
2. $\forall x \forall y(x < y \vee y < x \vee x = y)$
3. $\forall x \forall y \forall z([x < y \wedge y < z] \rightarrow [x < z])$
4. $\forall x(x < sx)$
5. $\forall x \forall y([y < sx] \rightarrow [y < x \vee y = x])$

Let σ be the following sentence

$$\forall x \exists y(x = sy).$$

Prove that $T \not\vdash \sigma$ and that $T \not\vdash \neg\sigma$.

Exercise 11.5. Let $\mathcal{L}$ and T be as defined in Exercise 11.4. Let φ be the following sentence:

$$\forall x \forall y([x < y] \rightarrow [sx < sy]).$$

Prove that $T \vdash \varphi$. (Hint: You only need to prove that $T \vdash \varphi$. You do not need to actually write down a deduction of φ from T. Use the meta-theorems given in class!)