Many exercises on vector fields.

Problem 10.1. (Found on a exam review of Prof. Greenfield's.) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle x^2y^3, -y\sqrt{x}\rangle$ and $\mathbf{r}(t) = (t^2, -t^3)$ for $0 \le t \le 1$.

Problem 10.2 (Review). Show that the vector field $\mathbf{H} = \frac{1}{r^2}\langle -y, x\rangle$ is not a gradient. Carefully explain your reasoning.

Problem 10.3 (Review). Compute the gradient of the function $f(x, y) = \arctan(y/x)$. Carefully explain why this does not contradict your answer to Problem 10.2, above.

Problem 10.4. (See p. 1138 of the book.) Find a positively oriented simple closed curve C in $\mathbb{R}^2$ which maximizes the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy.$$

Give a sketch. (Hint: Green's Theorem.)

Problem 10.5. Give a sketch (not necessarily to scale) of the *dipole field* $\mathbf{D} = \frac{1}{r^4}\langle 2xy, y^2 - x^2\rangle$. Compute $\nabla\times\mathbf{D}$ and $\nabla \cdot \mathbf{D}$. Is $\mathbf{D}$ independent of path?

Problem 10.6. Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^2$. Suppose that there is a constant $\lambda \in \mathbb{R}$ so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lambda$$

for every closed curve C. Show that $\mathbf{F}$ is conservative. Give a sketch.

Problem 10.7. Suppose that $a > 0$ is a constant. Prove that

$$\int_0^{2\pi} \frac{a\, dt}{a^2 \cos^2 t + \sin^2 t} = 2\pi.$$

What goes wrong with your explanation when $a < 0$?