These notes for MA3F2 are an adaptation of Brian Sanderson's notes, posted with his permission. The originals are available on the web at `maths.warwick.ac.uk/~bjs/MA3F2-page.html`.

Any errors below are due to me; please inform me of such via email at `s.schleimer@warwick.ac.uk`.

1 Basic definitions

1.1 Knots and their diagrams

A *knot* K is a smooth loop in three-space which does not self-intersect itself. A more precise definition might read:

> Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in the plane. A *knot* $K \subset \mathbb{R}^3$ is the image of a smooth embedding $f\colon S^1 \to \mathbb{R}^3$.

Since it is difficult to draw in $\mathbb{R}^3$, and easy to draw in the plane, we will visualize knots by projecting onto the xy–plane, and recording *crossing information.* So we define a *knot diagram* D to be a smooth loop in the plane which is allowed to transversely self-intersect at *crossings*. At each crossing there is exactly one overpassing and one underpassing arc. Here are a few examples:

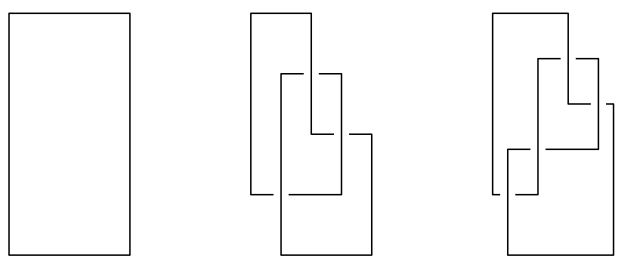

Figure 1: Diagrams of the unknot, the trefoil, and the figure eight. These are drawn as diagrams of polygonal knots.

The unknot is special: it is the only knot which is not knotted. Here are a few drawings which are *not* diagrams of knots:

Figure 2: Projections of an arc, a wild knot, a self-intersecting loop, and a loop with a triple point.

We can generalize the notion of a knot to include *links*: a link L is a collection of pairwise disjoint knots in $\mathbb{R}^3$.

Figure 3: Diagrams of the Hopf link, the Whitehead link, and the Borromean rings.

Notice that the Hopf and Whitehead links have two components while the Borromean rings have three. Any knot can be considered a link of one component.

We shall often abuse good sense and refer to a knot and the diagram of the knot interchangably. However they are not the same – this point will arise often in the course.

1.2 Orientations

Any knot has two possible *orientations*. A knot with an orientation is called an *oriented knot*. To orient a link simply orient the components. It follows that a link with n components has 2^n possible orientations.

Figure 4: All four of the orientations on the Hopf link.

If the knot or link L is oriented then every crossing of the diagram of L receives a handedness. The convention we will adopt in this class is shown in the following figure:

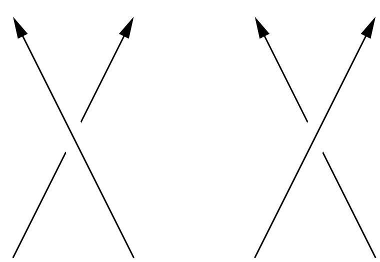

Figure 5: A left-handed crossing and a right-handed crossing.

This convention is similar to the right-hand rule in physics. One way to recall the right-handed crossing is to cross your right thumb over your right index finger. With orientation towards the ends of your fingers, this reproduces the right-handed crossing. Also, a collection of right-handed crossings glued end-to-end, as in the figure below, forms a right-handed screw.

Here is another way to remember the convention: in the standard diagram of the right-handed crossing the overpassing arc has positive slope. Right-handed crossings will be called *positive* while left-handed will be called *negative*. We will write $\text{sign}(c) = 1$ if c is a right-handed crossing and $\text{sign}(c) = -1$ if c is left-handed.

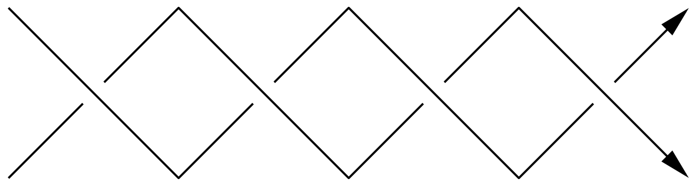

Figure 6: A sequence of right-handed crossings.

Definition 1.1. Suppose that D is a diagram of an oriented knot. The *writhe* of D is the integer

$$w(D) = \sum \mathrm{sign}(c)$$

where the sum is taken over the crossings c of the diagram.

Figure 7: The writhe of the right-handed trefoil, the Hopf link, and several unknots.

Exercise 1.2. Suppose that D is an oriented diagram of a knot (or link) and $-D$ is the diagram with opposite orientation. Prove that $w(-D) = w(D)$.

Exercise 1.3. Suppose that D is an oriented diagram and $\overline{D}$ is the mirror-image diagram. Prove that $w(\overline{D}) = -w(D)$.

Definition 1.4. Now suppose that D is an oriented diagram of a link with components $\{C_i\}_{i=1}^n$. We define the linking number of C_i and C_j, for $i \neq j$ by

$$\mathrm{lk}(C_i, C_j) = \frac{1}{2} \sum \mathrm{sign}(c)$$

where the sum is taken over the crossings c between C_i and C_j.

Remark 1.5. The linking number between two knots was first defined by Gauss, as the degree of a certain Gauss map. (Of course, he probably didn't call it that!)

Exercise 1.6. Suppose that $D = \bigsqcup C_i$ is a diagram. Prove that $\mathrm{lk}(C_i, C_j)$ is an integer.

The total linking number of the diagram D is:

$$\mathrm{lk}(D) = \sum_{i<j} \mathrm{lk}(C_i, C_j).$$

Here are several examples:

Figure 8: Orientations of the Hopf link, the Whitehead link, the Borromean rings, and a boring three-component link.

2 Isotopy

Links, as curves in three-space, are analytical objects. In this course, we wish to treat links combinatorially. To do that, we must decide when two links, perhaps differing analytically, are combinatorially the same.

To that end we say two links K and L are *isotopic* if there is a smooth deformation of $\mathbb{R}^3$ throwing K onto L. A more precise definition might read:

> Suppose that K is the image of $f\colon \bigsqcup S^1 \to \mathbb{R}^3$. The links K and L are isotopic if there is an smooth map $F\colon \mathbb{R}^3 \times I \to \mathbb{R}^3$ so that
>
> - for all $t \in I$ the map $F_t = F(\cdot, t)$ is a diffeomorphism,
> - $F_0 = \mathrm{Id}$, and
> - L is the image of $F_1 \circ f\colon S^1 \to \mathbb{R}^3$.

We have our first foundational result:

Theorem 2.1 (Reidemeister). *Every link K is isotopic to a link K' so that projection to the xy plane gives a diagram of K'.* □

An easy converse is:

Lemma 2.2. *Every four-valent graph in the plane, equipped with crossing information, is the diagram of some link. Futhermore, if two links have the same projection then the links are isotopic.*

Proof. To see the first: Suppose that D is such a graph. Draw the link K_D in $\mathbb{R}^3$ so that K_D agrees with D away from the crossings and do the natural thing at over and underpasses.

To see the second: if K projects to D then we may vertically isotope K to agree with K_D, the link constructed above. □

To restate, every link may be isotoped to have a diagram and every diagram comes from a link. However, the isotopies used above are very limited. We now must gain diagrammatic control over all isotopies.

Suppose that K and L are isotopic. The isotopy between them gives a family of links $\{K_t\}_{t \in I}$ where $K_t = F_t(K)$. Thus $K_0 = K$ and $K_1 = L$. There is a corresponding sequence of projections D_t. For a generic isotopy we expect all but finitely many of the

Figure 9: You cannot isotope a knot to the unknot just by "pulling tight". Above K_t is a knot for all $t \in I$. However, the motion cannot be generated by an isotopy of $\mathbb{R}^3$.

D_t to be diagrams. At a time t when D_t is not a diagram, we expect that $D_{t-\epsilon}$ and $D_{t-\epsilon}$ will differ in a simple fashion.

Figures 10, 11, 12, and 13 display the four *Reidemeister moves*: each move is the projection of a particularly simple isotopy.

Figure 10: R_0: any planar isotopy of the diagram.

In moves R_1, R_2, and R_3 only part of the diagram is shown, and the isotopy is assumed to leave the rest of the knot alone. We tacitly include the mirror image of each of these moves, as well.

Figure 11: R_1: the right and left twists. These are resolutions of the cusp.

If orientations are present then the moves are assumed to preserve them. There is another move that we shall often use:

Here is our second foundational result.

Theorem 2.3 (Reidemeister). *Suppose that K and L are links with diagrams D and E. Then K and L are isotopic if and only if there is a sequence of Reidemeister moves connecting D and E.* □

The forward direction is the hard one and is comparable to Theorem 2.1

Exercise 2.4. Check that the backwards direction of Theorem 2.3 follows from Theorem 2.1 and Lemma 2.2.

Exercise 2.5. Show that the R_∞ move can be obtained as via a sequence of the standard four moves.

Exercise 2.6. Show that the figure eight is isotopic to its mirror image. (Use a piece of string!) Now draw a sequence of Reidemeister moves to prove that the two knots are isotopic. (Hint: Exercise 2.5 may be useful.)

Exercise 2.7. The figure eight has two orientations. Are these isotopic? If so, provide a sequence of Reidemeister moves.

Figure 12: R_2: the finger move. These are resolutions of the self-tangency.

Figure 13: R_3: these are resolutions of the triple point.

Knots and their isotopies are essentially topological in nature while diagrams and Reidemeister moves are combinatorial. Thus Theorem 2.3 gives a combinatorial method of understanding topological objects! In summary:

> *any property of diagrams which is invariant under the Reidemeister moves is in fact a property of isotopy classes of links.*

3 Isotopy invariants

Now that we have a grasp of what it means for two knots to be isotopic we may state a major question of knot theory:

Question 3.1. Is there an algorithm that, given two knots K and L, decides if K is isotopic to L?

This may be specialized as follows:

Question 3.2. Is there an algorithm that, given a knot K, decides if K is isotopic to the unknot?

In fact, the answer in each case is "yes" due to work of Haken in the 1960's. Haken's work is three-dimensional in nature and focuses on the surfaces contained in the complement of the knot. There is another approach following from the Geometrization Theorem of Thurston (for Haken manifolds). The approach based on Thurston's work relies on finding hyperbolic structures via algebraic geometry.

There is another solution to the Unknotting Problem which is more in the spirit of this class:

Theorem 3.3 (Dunfield-Garoufalidis). *The A–polynomial detects the unknot.* □

Figure 14: R_∞: dragging an arc of the diagram past the point at infinity.

That is, the A–polynomial is an isotopy invariant and the value it takes on the unknot is different from the value it takes on any other knot. To actually compute the A–polynomial one must be given a diagram of the knot in question. That the A–polynomial is an isotopy invariant follows from the fact that it is unchanged by the Reidemeister moves. Let us return to more solid ground.

3.1 Linking number

Here is a first example of a link invariant, defined diagrammatically. Suppose that K is a link and D is the diagram of K. Define the *linking numbers* of K to be the linking numbers of D.

Corollary 3.4. *The linking numbers of K are an invariant of the isotopy class of K.*

Proof. It suffices to check that the Reidemeister moves do not change linking number. Suppose that D and D' are diagrams related by an R_i move.

When $i = 0$ the diagrams D and D' have identical combinatorics. Thus any invariant combinatorally defined in terms of a diagram is unaffected by the R_0 move.

When $i = 1$ the writhe of a component of the link may change, but all crossings between distinct components are unchanged.

When $i = 2$ or 3, we have a "proof-by-picture." □

Corollary 3.5. *The unlink and the Hopf link are not isotopic. The Hopf link and the Whitehead link are not isotopic.*

Remark 3.6. Notice that the writhe of a link is *not* a knot invariant. Every right R_1 move increases the writhe by one while every left R_1 move decreases the writhe by one.

Exercise 3.7. Provide a short proof that the unlink and the Hopf link are not isotopic. Think about how you would prove that the unlink and the Whitehead link are not isotopic.

3.2 Knot coloring

We take $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Let D be the diagram of a link. Let $\{\alpha_i\}_{i\in|D|}$ be the collection of arcs of D. (That is, when you draw D, the arc α_i can be drawn without lifting your pencil.) Here $|D|$ is the number of arcs of D. At each crossing c of D we see bits of three (perhaps non-distinct) arcs α_i, α_j, and α_k.

Definition 3.8. A link L can be *colored modulo n* if there is a function $x\colon |D| \to \mathbb{Z}_n$ so that, for every crossing c if the arcs α_i and α_j cross under α_k then

$$x_i + x_j = 2x_k \mod n.$$

That is, at every crossing c the overpassing arc α_k is "colored" with the average of the two underpassing arcs. (A small amount of care is required here – for example "average" is not defined when the modulus is even.) Notice that the constant function gives a coloring: we call any such a *trivial coloring*.

Lemma 3.9. *If x and y are colorings, modulo n, of D then so is $x + y$.* □

Lemma 3.10. *If D has a non-trivial coloring modulo n then for any $i \in |D|$ there is a non-trivial coloring modulo n of D, say $x\colon |D| \to \mathbb{Z}_n$, where $x_i = 0$.*

Proof. Let y be the given non-trivial coloring. Let $z\colon |D| \to \mathbb{Z}_n$ be the constant function so that $z_j = -y_i$ for all j. Then $x = y + z$ is the desired coloring. □

Figure 15: Examples of non-trivial colorings. The left is a coloring of the trefoil modulo 3 while the right is a coloring of the figure eight modulo 5.

Theorem 3.11. *A diagram D has a nontrivial coloring modulo n if and only if any diagram D' related to D via a single Reidemeister move also has a non-trivial coloring modulo n.*

Proof. It suffices to check that the Riedemeister moves preserve triviality of colorings. This is an easy case by case analysis. □

Corollary 3.12. *All colorings of knot diagrams isotopic to the unknot are trivial.* □

Example 3.13. Let D be the usual diagram of the trefoil. Since the diagram has a three-fold symmetry, we can start with any arc, call it α_0. By Lemma 3.10 we may assume that $x_0 = 0$. Choosing a variable $a \in \mathbb{Z}_n$ we may choose the labels $x_1 = a$ and $x_2 = -a$. Now all three arcs are labelled. The equation coming from the crossing where a is the overcrossing is $2a = 0 + (-a)$ and thus we find that $3a = 0$ modulo n.

It follows that there is an integer k so that $3a = kn$ in $\mathbb{Z}$. Since 3 is prime, we find that either k or n is a multiple of 3. Now, if 3 divides k then a is a multiple of n and the coloring we have found is trivial. We deduce that *every* modulo n coloring of trefoil is either trivial or has n divisible by three.

Example 3.14. Let D be the usual diagram of the figure eight knot. Label the central arc with a zero, and carry out a similar analysis. As argued in Example 3.13 since 5 is prime, any non-trivial modulo n coloring of the figure eight has n divisible by five.

Now, the unknot only has trivial colorings. It follows that the unknot, the trefoil, and the figure eight are none isotopic to the others.

Proposition 3.15. *A link diagram D has a non-trivial two coloring iff D has at least two components.*

Proof. Draw the four allowable colorings near a crossing and note that the two underpassing arcs receive the same label. Thus in a coloring modulo 2 every component receives the trivial coloring. This proves the forward direction.

If D has more than one component then color one of them with a zero and all of the others with a one. □

Definition 3.16. A link $L \subset \mathbb{R}^3$ is *split* if L is disjoint from the yz plane but meets both half-spaces $\{(x, y, z) \mid x > 0\}$ and $\{(x, y, z) \mid x < 0\}$. The diagram of a split link is called a split diagram.

A link is *splittable* if it is isotopic to a split link. Again, we use the same language for diagrams.

Proposition 3.17. *If D is splittable then D has non-trivial colorings in every modulus greater than one.*

Proof. Fix n. Perform Reidemeister moves to split the diagram. Now color the components in the positive half-space with a zero. Color the components in the negative half-space with a one. We are now done, by Theorem 3.11. □

Corollary 3.18. *The Borromean rings are not splittable.*

Proof. The Borromean rings cannot be non-trivially colored modulo three. □

Exercise 3.19. Show that the standard diagram of the Borromean rings cannot be 3–colored.

Figure 16: A four coloring of the Borromean rings.

Exercise 3.20. As in the examples above, find non-trivial colorings of the Whitehead link. Careful: you cannot divide by two in the ring $\mathbb{Z}_{2m}$.

4 Shadows and checkerboards

As we have seen above, a coloring of a diagram may be thought of as a collection of variables (one for each arc of the diagram). Every crossing now gives a relation among these variables. We'll call this the *crossing equation*. The examples you computed above all suggest that there is a dependence among the crossing equations. This is indeed the case: the proof is somewhat lengthy!