Exercise 5.8. Compute the determinant of the 5_1 knot (the cinefoil) and the twist knots. Notice that the first two twist knots are the trefoil and the figure eight.

6 Smith normal form and the coloring group

6.1 Smith normal form and Cramer's rule

We now find a method, more general than the one involving Cramer's rule, for finding non-trivial colorings.

Theorem 6.1. *The link L has a non-trivial coloring modulo $n > 1$ if and only if* $\gcd(n, \det(L)) > 1$.

We need to recall Smith normal form:

Lemma 6.2. *Given a $d \times d$ integer matrix A, as above, there are matrices R, C, B so that R, C are isomorphisms of $\mathbb{Z}^d$, B is diagonal, and $B = RAC$.*

It follows that $\det(A) = \det(B)$.

Proof of Lemma 6.2. Use row (and column) operations of the form

Type 1. $r_i \mapsto r_j$ and $r_j \mapsto -r_i$, and

Type 2. $r_i \mapsto r_i + ar_j$ for $i \neq j$ and $a \in \mathbb{Z}$.

Notice that operating on rows is equivalent to multiplying A on the left by a signed permutation matrix (1) or an elementary matrix (2). Operating on columns is equivalent to multipying on the right. Also, such matrices are isomorphisms of $\mathbb{Z}^d$.

Exercise 6.3. Show that type 1 operations can be obtained from three operations of type 2.

Let a_{ij} be a non-zero entry of A of smallest absolute value. Using type 1 operations move a_{ij} into the upper-left corner of A.

$$A \mapsto \begin{bmatrix} a_{ij} & * \\ * & * \end{bmatrix}$$

Now use type 2 operations to add or subtract row one (column one) from the other rows (columns) to decrease the absolute value of all other entries in the first row and column. Repeat the above steps until all entries in the first row and column, except for perhaps the upper-left entry, are zero. Note that this process must finish, as the absolute value of the upper-left entry only decreases.

Now the matrix has block form, with $d-1$ zeros in the first row and $d-1$ zeros in the first column.

$$\begin{bmatrix} b_1 & \overline{0} \\ \overline{0} & B_1 \end{bmatrix}$$

Here $\overline{0}$ is the zero row or column vector of the correct size. We now repeat the above process in the lower-right corner.

$$\begin{bmatrix} b_1 & 0 & \overline{0} \\ 0 & b_2 & \overline{0} \\ \overline{0} & \overline{0} & B_1 \end{bmatrix}$$

The process terminates with the desired diagonal matrix B. □

In fact, we may also arrange that the diagonal entries $b_1, b_2, \ldots, b_d$ of B have the property that $b_i | b_{i+1}$. We will not need this extra property.

Proof of Theorem 6.1. Since $B = RAC$ we have a square of maps:

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{A} & \mathbb{Z}^d \\ {\scriptstyle C}\uparrow & & \downarrow{\scriptstyle R} \\ \mathbb{Z}^d & \xrightarrow{B} & \mathbb{Z}^d \end{array}$$

where the vertical arrows are isomorphisms. Now, if we reduce modulo n we obtain a new square

$$\begin{array}{ccc} \mathbb{Z}_n^d & \xrightarrow{A_n} & \mathbb{Z}_n^d \\ {\scriptstyle C_n}\uparrow & & \downarrow{\scriptstyle R_n} \\ \mathbb{Z}_n^d & \xrightarrow{B_n} & \mathbb{Z}_n^d \end{array}$$

and the link L has a non-trivial coloring modulo n if and only if $\ker(A_n)$ is non-trivial. Since C_n is again an isomorphism, $C_n(\ker(B_n)) = \ker(A_n)$. Thus L may be so colored if and only if B_n has a kernel. Since B_n is diagonal, to have a kernel there is some $c \in \mathbb{Z}$ and some diagonal entry b_i so that $b_i c = 0 \mod n$ with $c \neq 0 \mod n$. This last happens if and only if $\gcd(n, b_i) > 1$. Since $|\prod b_i| = \det(L)$ the theorem is proven. □

Remark 6.4. The problem of finding colorings is now completely solved, in principle. The matrix B records which moduli have non-trivial colorings. To actually produce such it suffices to apply the matrix C_n to an element of the kernel of B_n.

In the above we restricted to diagrams without overcrossing components. For such diagrams the matrix A_+ is not square. However, we may always perform Reidemeister moves to "cure" the bad component. Then, since the link is splittable, Proposition 3.17 implies that L has non-trivial colorings in every modulus. Thus L falls into Case (i) above, and $\det(L) = 0$, as expected.

We now turn to several examples:

Exercise 6.5. Let $T(2,4)$ be the $(2,4)$–torus link. Let W be the Whitehead link. Show that $\det(T(2,4)) = 4$ while $\det(W) = 8$.

It follows that the coloring numbers cannot distinguish $T(2,4)$ from W in the same way we differentiated between the W and the Borromean rings. However, the determinant does distingush $T(2,4)$ and W.

As we have seen above, if a link is splittable then $\det(L) = 0$. The converse is false: the link shown in Figure 17 has a coloring over $\mathbb{Z}$ and so has determinant zero. (The link is "clearly" not isotopic to the unlink: can you provide a proof?)

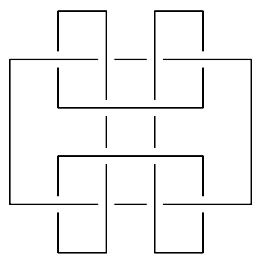

Figure 17: Not the unlink. Does anybody know a nice name for this link?

Finally, we give an example of a knot with $\det(K) = 1$. Define $P(p,q,r)$ to be the (p,q,r)–*pretzel* link. See Figure 18 for a picture of the $(4,-4,4)$-pretzel link.

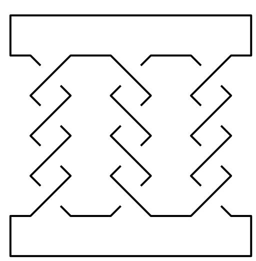

Figure 18: The (4, -4, 4)-pretzel link.

Exercise 6.6. Show that $P = P(-2,3,5)$ has determinant equal to one.

Exercise 6.7. Work out the determinant for general pretzel knots $P(p,q,r)$.

6.2 The coloring group

Definition 6.8. Suppose that D is a connected, reduced diagram without overcrossing components of a link L. Define $\mathrm{Col}(L)$ to be the abelian group with generators equal to the collection of all overcrossing arcs but one, and relations being the crossing equations

$$\mathrm{Col}(L) = \langle x_1, x_2, \ldots, x_d \mid x_k + x_k = x_i + x_j \rangle$$

where on the right the missing overcrossing arc is set to be zero.

Proposition 6.9. *The group* $\mathrm{Col}(L)$ *is an isotopy invariant.*

Exercise 6.10. Prove this. As usual, it suffices to check that $\mathrm{Col}(L)$ is unchanged, up to isomorphism, by the Reidemeister moves.

The coloring group $\mathrm{Col}(L)$ can be identified with the quotient group $\mathbb{Z}^d / A^T \mathbb{Z}^d$ where A^T is the transpose of the matrix obtained by deleting a row and column from the matrix A_+ of crossing equations. To see this recall that we had:

$$\{\text{solutions modulo } n\} \to \mathbb{Z}_n^d \to \mathbb{Z}_n^d.$$

The second map is A_n; the matrix A modulo n. The second term is generated by arcs, the third is generated by crossings, and the first is the kernel of A_n. Taking the transpose of A gives a "reversed" collection of groups:

$$\mathbb{Z}^d \to \mathbb{Z}^d \to \mathrm{Col}(L)$$

where the first map is A^T, the first term is generated by crossings, the second by arcs, and the third is the cokernel.

Proposition 6.11. *Suppose that L is a link. Let A be the martix obtained by deleting a row and column from the matrix A_+ of coloring equations. As in Lemma 6.2 we have a diagonal matrix $B = RAC$. Then* $\mathrm{Col}(L) \cong \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}$.

Proof. As $B = RAC$ it follows that $B^T = C^T A^T R^T$. Since B is diagonal, $B^T = B$, as well. Now consider the diagram of maps:

$$\begin{array}{ccccc} \mathbb{Z}^d & \xrightarrow{A^T} & \mathbb{Z}^d & \longrightarrow & \mathrm{Col}(L) \\ {\scriptstyle R^T}\uparrow & & \downarrow{\scriptstyle C^T} & & \downarrow ? \\ \mathbb{Z}^d & \xrightarrow{B^T} & \mathbb{Z}^d & \longrightarrow & \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}. \end{array}$$

That is, $\mathrm{Col}(L)$ is the cokernel of A^T and $\mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}$ is the cokernel of B^T. As argued in the proof of Theorem 6.1, since $B = RAC$ and R, C are isomorphisms, many invariants of A and B agree. In particular we can build an isomorphism between the cokernels of A^T and B^T. The argument is a classic "diagram chase": a more refined version is called the *five lemma* in algebra.

Suppose that $x \in \mathrm{Col}(L)$ is any element. Choose a preimage of x in $\mathbb{Z}^d$ say x', project x' down to $y' \in \mathbb{Z}^d$ via the map C^T and then forward to y in the cokernel of B^T. We define a map

$$\phi\colon \mathrm{Col}(L) \to \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}.$$

by sending x to y. Our first task is to show that ϕ is well-defined. So suppose that x'' is another preimage of x which projects to z' and then to z in $\mathrm{coker}(B^T)$. We must show that $y = z$.

Since x' and x'' are both preimages of x, by the definition of the cokernel, their difference $x' - x''$ lies in the image of A^T. Choose any preimage of $x' - x''$ and call that ξ. This projects, via the inverse of R^T, to an element ξ' which is carried forward by B^T. Since $B^T = C^T A^T R^T$ we find that

$$B^T(\xi') = C^T A^T R^T(\xi') = C^T A^T(\xi) = C^T(x' - x'') = y' - z'.$$

Since $y' - z'$ is in the image of B^T the element maps to zero in the cokernel. Thus $y - z = 0$, as desired.

Our second task is to prove that ϕ is an isomorphism. I leave this as an exercise. □

Since the order of $\mathbb{Z}_{b_1} \times \ldots \mathbb{Z}_{b_d}$ equals $\prod b_i$ (as long as all of the $b_i \neq 0$) we have:

Corollary 6.12. *If* $\det(L) \neq 0$ *then the group* $\mathrm{Col}(L)$ *has order* $\det(L)$*. If* $\det(L) = 0$ *then the group* $\mathrm{Col}(L)$ *is infinite.*

Corollary 6.13. *The determinant of a link is an isotopy invariant.*

Example 6.14. The coloring group is a stronger invariant than the determinant. For example, the knots 6_1 and 9_{46} both have determinant equal to 9. However, $\mathrm{Col}(6_1) = \mathbb{Z}_9$ while $\mathrm{Col}(9_{46}) = \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proposition 6.15. *Suppose that L is a link and B is the diagonal matrix defined above, with digaonal entries $b_1, \ldots, b_d$. Then the number of distinct colorings modulo n is*

$$\prod \gcd(b_i, n).$$

Proof. A coloring of the diagram D is a function x taking overcrossing arcs to elements of $\mathbb{Z}_n$ so that one distinguished arc is always taken to zero and so that the crossing equations are satisfied. Any such function gives a homomorphism $x\colon \mathrm{Col}(L) \to \mathbb{Z}_n$. The collection of such homomorphisms forms a group

$$\mathrm{Hom}(\mathrm{Col}(L), \mathbb{Z}_n).$$

The order of this group is the number of distinct colorings. Since $\mathrm{Col}(L) \cong \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}$ the conclusion follows from the identities

$$\mathrm{Hom}(\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z}_{\gcd(a,b)}$$

and

$$\mathrm{Hom}(\mathbb{Z}_a \times \mathbb{Z}_b, \mathbb{Z}_c) = \mathrm{Hom}(\mathbb{Z}_a, \mathbb{Z}_c) \times \mathrm{Hom}(\mathbb{Z}_b, \mathbb{Z}_c).$$

□

Example 6.16. The knot 6_1 has only three distinct colorings modulo 3 while the knot 9_{46} has nine.

Example 6.17. The coloring group can also distinguish the two-component unlink from the link shown in Figure 17, even though both have determinant zero. The coloring group of the unlink is $\mathbb{Z}$ while the other has $\mathrm{Col}(L) \cong \mathbb{Z} \times \mathbb{Z}_3^2$. Proposition 6.15 tells us that L has twenty-seven colorings modulo 3 while the unlink has only three.

Exercise 6.18. Compute the matrix A_+ for the link L shown in Figure 17. Compute the Smith normal form of the associated matrix A. Use this to check the claim made in Example 6.17.

Remark 6.19. Several times in class we have labelled one arc of a diagram with a zero, labelled some of the other arcs in an ad hoc fashion, and then deduced labels on all of the other arcs. (Occasionally we also find relations amongst the original labels.) This procedure can be thought of as computing the group $\mathrm{Col}(L)$.

Exercise 6.20. Reproduce the direct computation of $\mathrm{Col}(L)$, where L is the link shown in Figure 17, as performed in class.

Remark 6.21. We remark that up to seven crossings the coloring numbers suffice to distinguish all knots in the tables. However 8_2 and 8_3 both have coloring group isomorphic to $\mathbb{Z}_{17}$. In Section 8 we will discuss an invariant which distinguishes these knots.

Exercise 6.22. Check that 8_2 and 8_{17} have isomorphic coloring groups.

7 Mirrors, inversion, and codes

7.1 Mirrors

Let $L \subset \mathbb{R}^3$ and let $M \subset \mathbb{R}^3$ be an affine plane. That is, M need not contain the origin. The *mirror* of L, $m(L)$, is the link obtained by reflecting L through the plane M.

Proposition 7.1. *If L_0 and L_1 are isotopic then so are $m(L_0)$ and $m(L_1)$.*