The order of this group is the number of distinct colorings. Since $\mathrm{Col}(L) \cong \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_d}$ the conclusion follows from the identities

$$\mathrm{Hom}(\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z}_{\gcd(a,b)}$$

and

$$\mathrm{Hom}(\mathbb{Z}_a \times \mathbb{Z}_b, \mathbb{Z}_c) = \mathrm{Hom}(\mathbb{Z}_a, \mathbb{Z}_c) \times \mathrm{Hom}(\mathbb{Z}_b, \mathbb{Z}_c).$$

□

Example 6.16. The knot 6_1 has only three distinct colorings modulo 3 while the knot 9_{46} has nine.

Example 6.17. The coloring group can also distinguish the two-component unlink from the link shown in Figure 17, even though both have determinant zero. The coloring group of the unlink is $\mathbb{Z}$ while the other has $\mathrm{Col}(L) \cong \mathbb{Z} \times \mathbb{Z}_3^2$. Proposition 6.15 tells us that L has twenty-seven colorings modulo 3 while the unlink has only three.

Exercise 6.18. Compute the matrix A_+ for the link L shown in Figure 17. Compute the Smith normal form of the associated matrix A. Use this to check the claim made in Example 6.17.

Remark 6.19. Several times in class we have labelled one arc of a diagram with a zero, labelled some of the other arcs in an ad hoc fashion, and then deduced labels on all of the other arcs. (Occasionally we also find relations amongst the original labels.) This procedure can be thought of as computing the group $\mathrm{Col}(L)$.

Exercise 6.20. Reproduce the direct computation of $\mathrm{Col}(L)$, where L is the link shown in Figure 17, as performed in class.

Remark 6.21. We remark that up to seven crossings the coloring numbers suffice to distinguish all knots in the tables. However 8_2 and 8_3 both have coloring group isomorphic to $\mathbb{Z}_{17}$. In Section 8 we will discuss an invariant which distinguishes these knots.

Exercise 6.22. Check that 8_2 and 8_{17} have isomorphic coloring groups.

7 Mirrors, inversion, and codes

7.1 Mirrors

Let $L \subset \mathbb{R}^3$ and let $M \subset \mathbb{R}^3$ be an affine plane. That is, M need not contain the origin. The *mirror* of L, $m(L)$, is the link obtained by reflecting L through the plane M.

Proposition 7.1. *If L_0 and L_1 are isotopic then so are $m(L_0)$ and $m(L_1)$.*

Proof. Suppose that L_t, for $t \in [0,1]$ interpolates between L_0 and L_1. Then $m(L_t)$ interpolates between $m(L_0)$ and $m(L_1)$. □

Proposition 7.2. *The isotopy class of the link $m(L)$ is independent of the choice of plane M.*

Proof. Suppose that M, M' are a pair of affine planes and $m(L), m'(L)$ are the links obtained by reflecting L in each. Consider the homeomorphism $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ which is gotten by reflecting first in M and then in M'. This is a orientation preserving isometry of $\mathbb{R}^3$. If M and M' are parallel then ϕ is a translation. If M and M' meet then ϕ is a rotation about the line of intersection, by angle twice the dihedral angle between M and M'. In either case, there is an isotopy (motion of $\mathbb{R}^3$) taking ϕ to the identity map. Since ϕ throws $m(L)$ onto $m'(L)$ it follows that $m'(L)$ is isotopic to $m(L)$. □

Proposition 7.3. *A diagram for $m(L)$ is obtained by switching all of the crossings of a diagram for L.*

Proof. As in Lemma 2.2, we draw the diagram for L on the xy–plane in $\mathbb{R}^3$. We lift the overcrossings slightly into $\{(x,y,z) \mid z > 0\}$ and push the undercrossings slightly into $\{(x,y,z) \mid z < 0\}$. Reflecting in the xy–plane now gives the desired conclusion. □

Example 7.4. The right and left trefoils are mirror images of each other. In general, every crossing of $m(L)$ has sign the negative of the sign of the corresponding crossing in L.

Definition 7.5. A link is *achiral* (also: *amphichiral*) if it is isotopic to its mirror image. If not, it is chiral.

Exercise 7.6. If you haven't already done so, check that the figure eight knot (4_1) is achiral.

Remark 7.7. The knot tables (for example `http://www.math.toronto.edu/~drorbn/KAtlas/Knots/` or `http://www.indiana.edu/~knotinfo/`) only display one of K and $m(K)$. Of the 35 knots with at most eight crossings only 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, and 8_{18} are achiral. Classically, it was considered difficult to determine the chirality of a knot. Dehn, in a 1914 paper, used the *fundamental group* of the *knot complement* to show that the left and right trefoils are not isotopic.

Proposition 7.8. *The links L and $m(L)$ have isomorphic coloring groups.*

It follows that the coloring group, and hence the determinant, cannot distinguish the isotopy classes of L and $m(L)$.

Proof of Proposition 7.8. Use a mirror plane which is perpendicular to the xy–plane. This gives a bijection between the arcs of the two diagrams. Since the coloring relations are also preserved, this bijection induces the desired isomorphism of groups. □

7.2 Invertibility

Definition 7.9. Let L be an oriented link and let $r(L)$ denote L with all orientations reversed. If L and $r(L)$ are isotopic as oriented links then L is *invertible*. (For the isotopy we require that all components are sent to themselves, but with orientation reversed.)

Exercise 7.10. Show that all pretzel knots are invertible. Are all pretzel links invertible?

We remark that the only knots, up to nine crossings, which cannot be inverted are 8_{17}, 9_{32}, and 9_{33}. This is not easy to prove! Among the nine crossing knots only 9_{32} is neither invertible nor achrial. That is, if $K = 9_{32}$ then $K, r(K), m(K), rm(K)$ are four distinct isotopy classes (of oriented knots).

7.3 Knot codes

Recall now that a diagram is alternating if every overcrossing arc crosses over exactly one crossing. (This is the best definition in the course!)

Proposition 7.11. *Suppose that S is a shadow (diagram without crossing informations) with only one component. Then S represents (up to reflection) a unique inoriented alternating knot.*

Proof. Orient the component and introduce crossings by walking along the shadow and alternating between under and overcrossings. To prove that no conflict arises choose a checherboard coloring of the diagram and notice that when black (white) is on the left we go under (over) at the next crossing. ☐

Remark 7.12. Among the eight or fewer crossing knots, only 8_{19}, 8_{20}, and 8_{21} do not admit alternating diagrams. (As a challange: how could we show that a knot K has *no* alternating diagrams?) However, this trend probably does not continue; Hoste *et al*[1998] conjecture that the percentage of knots with an alternating diagram goes to zero exponentially with crossing number.

Given a shadow with one component, a point on the shadow, and an orientation we may walk around the component numbering the crossings we come to. Every crossing receives exactly two numbers.

Exercise 7.13 (Hard.). Prove that every crossing receives one odd and one even number.

So every shadow gives a function

$$f\colon \{\text{odd numbers}\} \to \{\text{even numbers}\}.$$

We write this as a sequence $[f(1), f(3), f(5), \ldots]$. This sequence determines the shadow (up to R_0 and R_∞ moves) and so, by Proposition 7.11, determines an alternating knot. For example, the trefoil is represented by the sequence $[4, 6, 2]$. Now, different choices of shadow for a single isotopy class of knot may give different sequences. So we say that the lexicographically first such sequence is the *code* of the knot. Finally, when the diagram is not alternating, we decorate the sequence with minus signs to indicate "wrong" crossings. For example, the sequence $[6, 8^-, 2, 4^-]$ represents a non-alternating diagram of the trefoil.

Exercise 7.14. The knots 6_1, 6_2, 6_3, have codes

$$[4, 8, 12, 10, 2, 6] \quad [4, 8, 10, 12, 2, 6] \quad [4, 8, 10, 2, 12, 6]$$

respectively. Draw these and check that they are isotopic to the standard diagrams.

Exercise 7.15. Compute the codes for the granny and reef knots, shown in Figure 19.

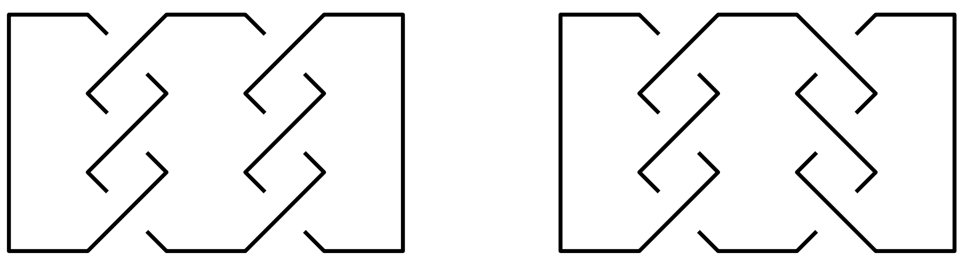

Figure 19: The granny knot is the *connect sum* of two right trefoils. The reef knot is the connect sum of a right with a left trefoil.

Exercise 7.16. Prove that there are at most $2^n \cdot n!$ knots, up to isotopy, with n or fewer crossings.

8 The Alexander Polynomial

8.1 The definition

Above, we studied colorings using labels from the group $\mathbb{Z}_n$. One way of defining a coloring is to view it as a function from the set of overcrossing arcs to $\mathbb{Z}_n$ satisfying the crossing relations. The relations must be carefully choosen – this done we find that the coloring number (that is, the modulus n) is an isotopy invariant of the knot or link.

The Alexander polynomial may be defined in a similar manner. We map the overcrossing arcs to the ring $\mathbb{Z}[t, t^{-1}]$, modulo a Laurent polynomial $\Delta(t)$. We now choose

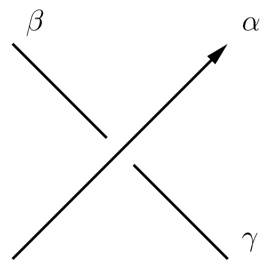

Figure 20: The names of the arcs at the overcrossing.

the crossing relations: Orient the components of the diagram. Fix attention on an overcrossing where α is the overcrossing arc, β is the arc to the left of α (facing in the direction of the orientation of α), and γ is the arc to the right of α. See Figure 20.

Suppose that α is labelled by the Laurent polynomial $a(t)$, β is labelled by $b(t)$, and γ is labelled by $c(t)$. At the crossing we impose the relation:

$$(1-t)a + tb - c = 0 \mod \Delta.$$

Note that the orientation on the understrand is not used.

Remark 8.1. Note that when $t = -1$ the above reduces to $2a - b - c = 0 \mod \Delta(-1)$, which is identical to the crossing crossing equation for colorings modulo $\Delta(-1)$.

To find such a labelling, as before, we consider A_+, the matrix of crossing equations. This time the entries in A_+ are polynomials in t. After deleting a column (eg, setting one label equal to zero) and deleting one row (eg, ignoring one crossing equation) we arrive at the matrix A. The determinant of this matrix is defined to be $\Delta_L(t)$, the *Alexander polynomial*. This will be an isotopy invariant more powerful than the determinant.

Example 8.2. Since the figure eight is alternating, we may order the arcs so that the arc α_i runs over the i^{th} crossing. This is carried out in Figure 21. This gives the matrix:

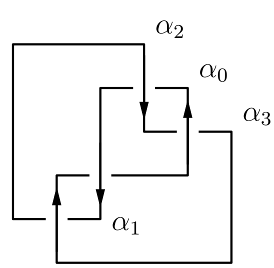

Figure 21: The figure eight knot, labelled and oriented.

$$A_+ = \begin{bmatrix} 1-t & 0 & t & -1 \\ t & 1-t & 0 & -1 \\ t & -1 & 1-t & 0 \\ 0 & -1 & t & 1-t \end{bmatrix}.$$

Deleting the first row and column gives

$$A = \begin{bmatrix} 1-t & 0 & -1 \\ -1 & 1-t & 0 \\ -1 & t & 1-t \end{bmatrix}$$

and this has determinant

$$(1-t)^3 + t - (1-t) = -t + 3t^2 - t^3 = \Delta_K(t).$$

(A common normalization is to multiply by $\pm t^n$ to make the first term positive and make the polynomial symmetric in t and $1/t$. For the figure eight knot we then have $\Delta_K(t) = t - 3 + t^{-1}$.) Cramer's rule, as in the discussion preceding Exercise 5.3, suggests that we take polynomials equal to:

$$x_0 = 0, \quad x_1 = (1-t)^2, \quad x_2 = 1-t, \quad x_3 = 1-2t.$$

Then we may check that

$$\begin{aligned} \Delta_K(t) &= (1-t)x_1 + 0 \cdot x_2 - x_3 \\ &= (1-t)(1-t)^2 - (1-2t) \\ &= 1 - 3t + 3t^2 - t^3 - 1 + 2t. \end{aligned}$$

Finally, we check the crossing equations (that is, check that $A_+ \cdot x = 0 \mod \Delta_K(t)$):

$$\begin{aligned} (1-t)x_0 + tx_2 - x_3 &= t(1-t) - 1 + 2t = -1 + 3t - t^2, \\ tx_0 + (1-t)x_1 - x_3 &= (1-t)^3 - (1-2t) = -t + 3t^2 - t^3, \\ tx_0 - x_1 + (1-t)x_2 &= -(1-t)^2 + (1-t)(1-t) = 0, \\ -x_1 + tx_2 + (1-t)x_3 &= -(1-t)^2 + t(1-t) + (1-t)(1-2t) = 0. \end{aligned}$$

Notice that one of the equations only sums to $\Delta_K(t)$ up to multiplication by $\pm t^n$. Also, none of the columns A_+ sum to zero. We will show that this can be remedied, below.

Exercise 8.3. Carry out the above for the trefoil knot, T. You should find that $\Delta_T(t) = t - 1 + t^{-1}$, up to multiplication by units in the ring $\mathbb{Z}[t, t^{-1}]$.

To arrange that each column of A_+ sums to zero, we will need a refined version of the checkerboard construction used in the proof of Proposition 4.11.

8.2 Winding number

Suppose that S is the shadow of an oriented diagram D. Isotope S slightly to put S in general position. For every region R of S we define a *winding number* $w(R)$ as follows: pick a point $x \in R$ and pick a ray L emanating from x, which is transverse to S. Every intersection of S and L receives a sign: $+1$ if the arc crosses L from right to left and -1 if the arc crosses L from left to right. Now define $w(R)$ to be the sum of these signs.

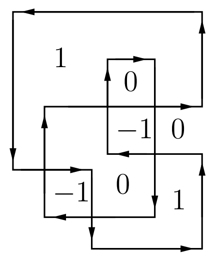

Figure 22: The winding number of the curve about each region is shown.

Exercise 8.4. Show that $w(R)$, the winding number of the oriented shadow around the region R, is well defined. Show that if R, R' are adjacent then $w(R) = w(R') \pm 1$. Show that $w(R) = e(R) \mod 2$ where $e(R)$ is the parity of the region (defined in the proof of Lemma 4.6).

8.3 Row sum

We can now justify the given definition of the Alexander polynomial. First of all, we consider the labelling to be a map from the set of overcrossing arcs to the ring $\mathbb{Z}[t, t^{-1}]/\Delta(t)$. As with Lemma 3.10, by subtracting a constant labelling we may assume that any one arc is sent to zero. Again, this corresponds to deleting a column from A_+.

To obtain a version of Proposition 4.11 for the Alexander polynomial: Fix an oriented diagram D which is connected, reduced, and has no overcrossing component. Label the regions of the diagram with their winding number.

Recall that G_D is the dual graph. Orient the edges of G_D so that all edges point from regions with lower winding number to regions with higher. Orient all quadrilaterals of $\mathbb{R}^2 \smallsetminus G_D$ using the counterclockwise convention.

Now, the crossings of D and the quadrilaterals of G_D are in one-to-one correspondence. Each quadrilateral gives an equation as follows: for each variable corresponding to a side of the quadrilateral multiply it by t^w where w is the winding number pointed at by the side. Mutiply the result by -1 if the orientation of the quadrilateral and the side disagree. Add the four resulting monomials.

For every crossing we find the sum equals

$$\pm t^n[(1-t)a + tb - c].$$

As with Proposition 4.11 with the above choices of crossing equation the sum of the crossing equations is zero.

8.4 Consequences

It is possible to give a proof that $\Delta_L(t)$ is an isotopy invariant of the link L, by examining the effect of the Reidemeister moves on the matrix A_+. However, we will content ourselves with a few consequences of the definition.

Proposition 8.5.

- $|\Delta_L(-1)| = \det(L)$.
- $\Delta_L(t) = \Delta_{r(L)}(t^{-1}) = \Delta_{m(L)}(t^{-1})$, *up to multiplication by units in the ring* $\mathbb{Z}[t, t^{-1}]$.

Proof. The first is the content of Remark 8.1. For the second, after reflection or inversion we find that the arc γ is now to the left of α. It follows that the crossing equation changes from

$$(1-t)a + tb - c = 0$$

to

$$(1-t)a' - b' + tc' = 0.$$

Now, if in A_+^K, we replace t everywhere by $1/t$ we find

$$(1 - 1/t)a(1/t) + 1/t \cdot b(1/t) - c(1/t) = 0.$$

Multiplying by $-t$ gives

$$(1-t)a(1/t) - b(1/t) + tc(1/t) = 0.$$

Thus the matrix $(-t)A_+^K(1/t)$ is identical to the matrix $A_+^{m(K)}(t) = A_+^{r(K)}(t)$ (possibly after multiplying some rows by units in the ring $\mathbb{Z}[t, t^{-1}]$). The conclusion follows. □

Remark 8.6. In fact, if K is a knot then $\Delta_K(t) = \Delta_K(t^{-1})$. See Rolfsen, pages 207-208, for a proof. It follows that the Alexander polynomial cannot distinguish a knot and its mirror image. So, even though Δ_K is much stronger than the determinant, we still cannot prove that the left and right trefoils are not isotopic.

Proposition 8.7. *If L is a splittable link then $\Delta_L(t) = 0$.*

Proof. Suppose that D is a split diagram for L. Let C and C' be the two subdiagrams. Then each of these has its own matrix of crossing equations, say B_+ and B'_+. Thus A_+ has block form:

$$A_+ = \begin{bmatrix} B_+ & 0 \\ 0 & B'_+ \end{bmatrix}.$$

Note that $\det(B_+) = \det(B'_+) = 0$. So, deleting the first row and column from A_+ gives

$$A = \begin{bmatrix} B & 0 \\ 0 & B'_+ \end{bmatrix}.$$

It follows that $\det(A) = \det(B) \cdot \det(B'_+) = \Delta_C(t) \cdot 0 = 0$, as desired. □

Remark 8.8. Note that the above proof, when it assumes that the diagram D is split, actually uses the unproven fact that Δ_L is an isotopy invariant.

8.5 Connect sums of knots

Suppose that K and L are oriented knots. Let D and E be diagrams of K and L respectively, where D lives in the left half-plane $\{(x, y) \mid x < 0\}$ and where E lives in the right.

We may assume, possibly after an R_1 move, that each of D and E has an outermost arc which is oriented in the counterclockwise fashion. Connect these arcs by an embedded path α which meets D and E each in exactly one end point. Remove a small arc from each of D and E (about the endpoints of α) and double α. Gluing everything together gives a diagram which we denote $D\#E$. Let $K\#L$ be any knot which has diagram $D\#E$. We call $K\#L$ the *connected sum* of the oriented knots K and L.

Theorem 8.9.

- *The knot $K\#L$ is well-defined up to isotopy and depends only on the isotopy classes of K and L.*
- *$K\#L$ is isotopic to $L\#K$.*
- *$K\#(L\#M)$ is isotopic to $(K\#L)\#M$.*
- *For the unknot U we find that $U\#K$ is isotopic to K.*

Proof. Begin with the first claim: Consider the diagrams D and E and the arc α. If we use a different arc α', perhaps connecting different outermost arcs of D and E, then we shrink E to be very small. Reel in the small copy of E along the arc α'. Now move the small copy of E along D until it is next to the correct outermost arc. Now reel E back out and enlarge. If necessary we can also shrink D, reel it into E, and so on.

Now, if D and D' are isotopic, then we can prove that $D\#E$ is isotopic to $D'\#E$. Shrink E, reel it in, and apply the isotopy that throws D onto D'. Now unreel E and enlarge.

The last three claims all have simple diagrammatic proofs, using the techniques of the first claim. □

Definition 8.10. A knot K is *prime* if whenever K is isotopic to $L\#M$ we find that one of L or M is isotopic to the unknot.

Note that the knot tables only record prime knots. As remarked above in the caption for Figure 19 the granny and reef knots are connect sums of trefoils. These two knots are obviously not isotopic, but the Alexander polynomial (and thus the coloring numbers, etc) is not powerful enough to see this:

Theorem 8.11. $\Delta_{K\#L}(t) = \Delta_K(t) \cdot \Delta_L(t)$, *up to multiplication by units of* $\mathbb{Z}[t, t^{-1}]$.

Since the right and left trefoils have the same Alexander polynomial, so do the reef and granny knots.

Proof of Theorem 8.11. Suppose that D and E are diagrams of K and L. Let D' and E' be the diagrams after a R_1 move. Let d_0 be the new crossing in D' and let e_m be the new crossing in E'. Let x_0 be the arc of D' crossing over d_0 and let y_0 be the arc of E' crossing over e_m.

Let A_+ and B_+ be the matrices of crossing equations for D' and E'. Define A to be the matrix obtained by deleting the d_0–row and x_0–column from A_+. Define B to the matrix obtained by deleting the e_m–row and y_0–column from B_+.

Now, $C = D'\#E'$ is the connect sum of diagrams, as described above. (This is the same as taking the connect sum $D\#E$ and performing an R_2 move between the two copies of α.) Here we find that the arcs x_0 and y_0 have been merged and there is a new undercrossing arc z. Take the matrix of crossing equations for C and delete the row corresponding to the crossing d_0 and the column for $x_0 = y_0$. Then the result has the form:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & v & t \end{bmatrix}$$

where $v = (0, \ldots, 0, -1)$ is a vector of the correct length. This matrix given has lower triangular block form and so its determinant gives the desired result. □