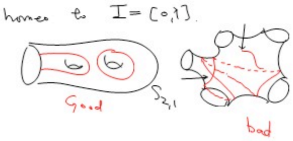
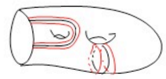
Lecture 2: Loops and arcs.

Def: A loop (curve) is a 1-submanifold in S homeo. to S^1. An arc is a 1-sub.— homeo to $I = [0,1]$.



So arcs and loops have product neighborhoods.

Picture



I.e $N(\alpha) \cong I \times \alpha \hookrightarrow S$ so that $\frac{1}{2} \times \alpha$ is sent to α via the projection onto 2^{nd} factor.

Def: If $\alpha \subset S$ then S_α = S cut along α is $S \setminus \mathring{N}(\alpha)$ = S minus the interior of $N(\alpha)$.

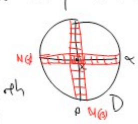
Recall: A cut system is a collection of loops $A = \{\alpha_1, \alpha_2, \ldots, \alpha_g\}$ so that S_A is planar.

Def: $\alpha \subset S$ is separating if S_α is not connected.

Exercise: Prove S is planar iff every loop $\alpha \subset S$ is separating.

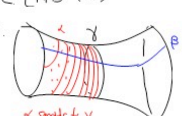
[Hints available via email.]

Def: $\alpha, \beta \subset S$ are transverse iff for every $x \in \alpha \cap \beta$ there $\exists$ a disk neighborhood of x and products $N(\alpha), N(\beta)$ so that $D \cap N(\alpha) \cap N(\beta)$ is a square in both $N(\alpha)$ and in $N(\beta)$



[I.e of form $I \times J \subseteq I \times \alpha$, J an interval and $I \times K \subseteq I \times \beta$, K " " .]

Exercise: If α, β are transverse then $|\alpha \cap \beta|$ = finite [NB this requires compactness.



Exercise:

$\alpha \subset S$ is nonsep iff $\exists \beta \subset S$ so that $\alpha \cap \beta$ is exactly one transverse intersection.

Exercise: Prove that Def ② + Def ③ of genus are equivalent

Reading Exercise: Epstein's paper (1966)

Prove the 2-Dim'l Poincaré Conj:

If $\pi_1(S) = \mathbb{1}$ then $S = \mathbb{D}^2$ or S^2

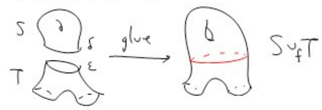
In summary: A surface is determined, up to homeomorphism, by g=genus(S), $b = |\partial S|$.

— [Class. Thm.] —

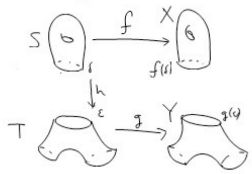
Want to classify curves on surfaces up to homeomorphism of the surface.

But first: Gluing,

Suppose S, T are surfaces, $\delta \subset \partial S$, $\varepsilon \subset \partial T$
Pick $f: \delta \to \varepsilon$



Glueing homeomorphisms is also allowed:



$R = S \cup_h T$

$Z = X \underset{f^{-1}hg}{\cup} Y$

<u>Define</u>: $F: R \longrightarrow Z$ via
$$F(x) = \begin{cases} f(x), & x \in S \\ g(x), & x \in T \end{cases}$$
The glued homeomorphism

Lots of homeomorphisms

<u>Exercise</u>: $\forall S$ there is a homeo

$\tau: S \to S$ s.t. $* \ \tau^2 = \mathbb{1}_S$
$*$ τ fixes every $\delta \subset \partial S$ setwise
$*$ $\tau|\delta$ is a reflection
[τ reverses orient. of S.]

Eg

$D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$

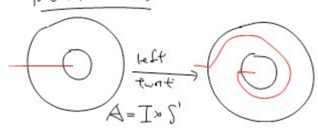
Consider $z \longmapsto \bar{z}$

Hint: induct on g, b. use gluing idea.

Connect sum useful

Call τ a <u>reflection</u>.

Dehn twists

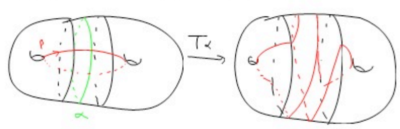


$A = I \times S^1$

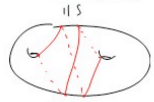
$$T_A(t, \theta) = (t, \theta + 2\pi t)$$

the inverse is a <u>right twist</u>. [NB: This depends on the orientation of A]

<u>Picture</u>: If $\alpha \subset S$, $A \cong N(\alpha)$



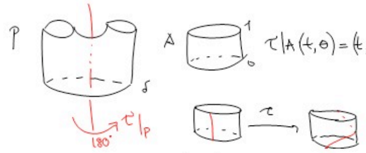
$$T_\alpha(x) = \begin{cases} x, & x \notin N(\alpha) \\ T_A(x), & x \in N(\alpha) \cong A \end{cases}$$



$T_\alpha(p)$

Half-twists:

Let $\tau|_P$ be 180° rotation of $P = S_{0,3}$



$\tau|A(t, \theta) = (t, \theta + \pi t)$

Glue to get a half twist: