<u>Reminder</u>: Homework to
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Register via email:
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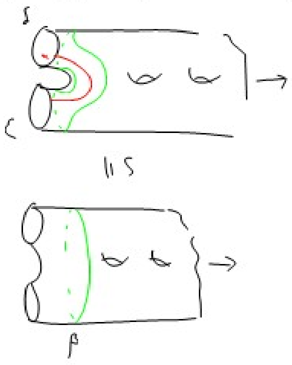
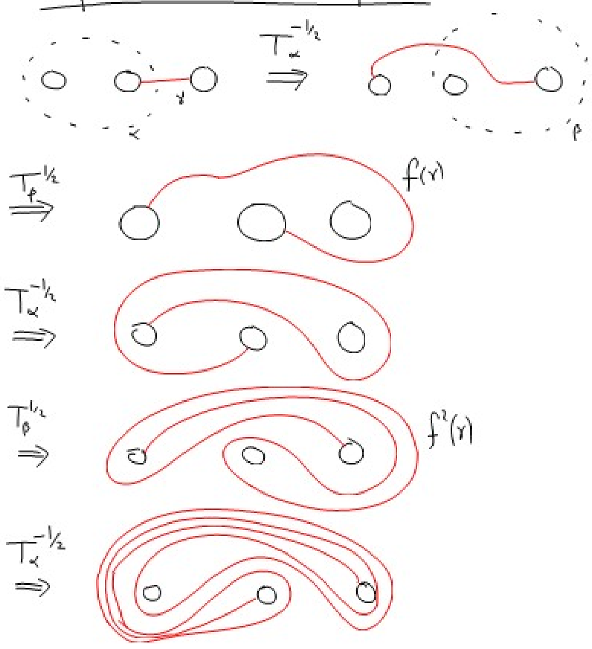
<u>Notation</u>. If $\alpha \subset S$ is an arc connecting $\delta, \varepsilon \subset \partial S$ then let P_α the the neigh of $\delta \cup \varepsilon \cup \alpha$

The <u>halftwist</u> in P_α is denoted by $T_\beta^{1/2}$



<u>Exercise</u>: For any $\delta, \varepsilon \subset \partial S$ there is a homeo $f: S \to S$ s.t. $f(\delta) = \varepsilon$

<u>Simple arcs can be complicated</u>



Define $f = T_\beta^{1/2} \cdot T_\alpha^{-1/2}$. <u>Exercise</u>: Compute $f^{10}(\gamma)$.

[Hint: You must find a way to organize all of the strands of $f^k(\gamma)$.]

<u>Easier</u>: let $g = T_\beta^{1/2} \cdot T_\alpha^{1/2}$. Compute $g^{10}(\gamma)$

<u>Explain</u> why this is easier,

[Hint: Thurston's Painting in Evans, Berkeley]

<u>Classification of arcs/loops on surfaces</u>.

<u>Topological</u>:

<u>Exercise</u>. If $\alpha, \beta \subset S$ are non-separating loops in S Then $\exists f: S \to S$ (homeo) st. $f(\alpha) = \beta$

[<u>similarly</u>. If $\alpha, \beta \subset S$ are arcs connecting distinct comp'ts of ∂S then $\exists$ homeo as before]

<u>Def</u>: $\mathrm{Homeo}(S) = \{ f: S \to S \mid f \text{ homeo} \}$

<u>Hint</u>: Use cutting, gluing, classification thm, half-twists, reflections.

<u>Exer</u>: Up to $\mathrm{Homeo}(S)$ there are only finitely many <u>multi curves</u>: $\{\alpha_1, \dots, \alpha_n\}$ with α_i's being disjoint and non-parallel.
= non-isotopic.

<u>Def</u>: $f, g \in \mathrm{Homeo}(S)$ are <u>isotopic</u> if $\exists$
$F: S \times I \to S$ s.t. (*) $F_t \in \mathrm{Homeo}(S)$
(*) $F_0 = f$, $F_1 = g$.

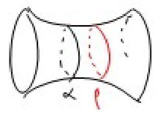
[Say $f \simeq g$]

<u>Def</u>: $\alpha, \beta \subset S$ are <u>isotopic</u> if $\exists f \in \mathrm{Homeo}_0(S)$ s.t. $f(\alpha) = \beta$

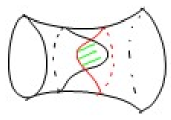
<u>Def</u>: $\mathrm{Homeo}_0(S) = \{ f \in \mathrm{Homeo}(S) \mid f \simeq \mathrm{Id}_S \}$

<u>Ex</u>: $\mathrm{Homeo}_0(S) \triangleleft \mathrm{Homeo}(S)$.

Pictures:



Cobound annulus $\Rightarrow$ isotopic



Isotopic after
1/ Remove bigon
2/ isotope across product annulus.

More defs: $\mathrm{Homeo}_+(S) = \{ f \mid f \text{ preserves orient} \}$

$\mathrm{Homeo}^{\partial}(S) = \{ f \mid f \text{ preserves compts of } \partial S \text{ each preserved setwise} \}$

$\mathrm{Homeo}(S, \partial) = \{ f \mid f|\partial S = \mathrm{Id}_{\partial S} \}$

Main def: $\mathcal{MCG}(S) \equiv \dfrac{\mathrm{Homeo}(S)}{\mathrm{Homeo}_0(S)}$

$\mathcal{S}(S) = \{ \text{essential, non-peripheral loops in } S \} / \simeq$

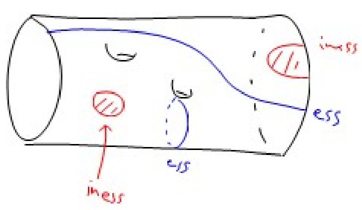
That is: $[\alpha]$ = isotopy class of α

if $\alpha, \beta \in [\alpha]$ then $\alpha \simeq \beta$ (isotopic)

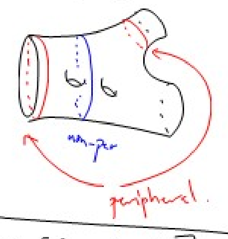
That is: Classifying elements of $\mathcal{S}$ is much harder than classifying elts of $\mathcal{S}(S) / \mathrm{Homeo}(S)$

Def: A separating loop or arc $\alpha \subset S$ is *inessential* if S_α contains a disk $[S_\alpha \equiv S \setminus \mathring{N}(\alpha)]$

Picture



Def: A separating loop $\alpha \subset S$ is *peripheral* if S_α contains an annulus



Lemma [Gugenheim 1953]
if α, β are iness loops then $\alpha \simeq \beta$.

[All disks $\mathbb{D}^2 \hookrightarrow S$ are isotopic [if orient pres]]

Thm: All loops in $\mathbb{D}^2$ are iness.
Pf: Jordan curve Thm.

That is: $\mathcal{S}(\mathbb{D}^2) = \emptyset$.

Rmk: The same holds for arcs in $\mathbb{D}^2$

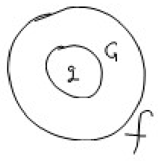
Claim: $MCG(\mathbb{D}^2) = \mathbb{Z}/2\mathbb{Z} \equiv \langle \text{reflection} \rangle$

Need: $\mathrm{Homeo}^+(S^1)$ deformation retracts to $SO(2)$

Exercise: Prove this.

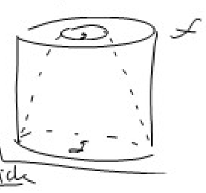
Suppose $g \in \mathrm{Homeo}^+(\mathbb{D}^2)$. will isotope to $\mathrm{Id}_{\mathbb{D}^2}$ in stages

Stage 1: Isotope in a collar to arrange $f \simeq g$. $f|\partial\mathbb{D}^2 = \mathrm{Id}_\partial$. **Picture**



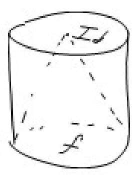
Here $G : \partial\mathbb{D} \times I \to \partial\mathbb{D}$ is the given isotopy $g|\partial \simeq \mathrm{Id}_\partial$

I.e. $f \simeq g$ "looks like"



Now for the *Alexander trick*

Trick: $\mathrm{Homeo}(\mathbb{D}^2, \partial)$ deformation retracts to Id.



$f \in \mathrm{Homeo}(\mathbb{D}^2, \partial)$

Exercise: $MCG(S^2) = \langle \text{reflection} \rangle$

Reading exercise
$\mathrm{Diffeo}^+(S^2)$ def. retracts to $SO(3)$
[Smale's Thm].