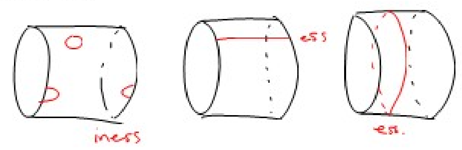
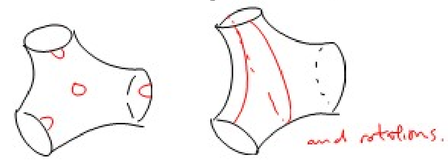
Exercise: Classify loops and arcs in $\mathbb{A}^2$ and P.

Ans



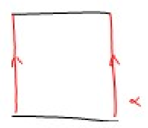
That is: there is a unique ess loop in $\mathbb{A}^2$.

P:

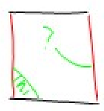


All ess. loops are peripheral.

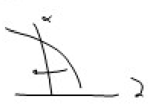
Idea: Cut $\mathbb{A}^2$ along an ess arc

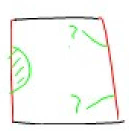


A cut along α. Now to classify loops and arcs in A use the classification in $\mathbb{D}^2$.

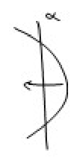


triangles





bigons

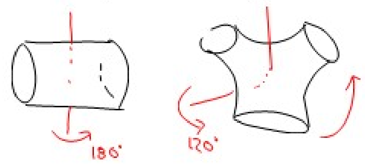


[Notice that the bigon is the "double" of the triangle.

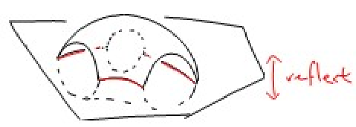
Consequence: Exercise:

$$MCG(A) = \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$MCG(P) = \Sigma_3 \times \mathbb{Z}_2$$

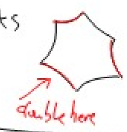
} permutations of ∂S $\times$ $\langle$reflection$\rangle$



Picture:



Doubling a hexagon gives a pants



Recall: $MCG(S) = \dfrac{Homeo(S)}{Homeo_0(S)}$

$MCG(S, \partial) = \dfrac{Homeo(S, \partial)}{Homeo_0(S, \partial)}$ (i.e. all homeos, isotopies fix ∂ pointwise)

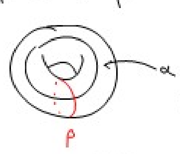
Def: Geom. intersection #

$$i(\alpha, \beta) = \min \left\{ |\alpha' \cap \beta'| \;\middle|\; \alpha' \simeq \alpha, \beta' \simeq \beta \right\}.$$

This is an isotopy invariant, by construction.

To compute $i(\alpha, \beta)$ seems difficult!

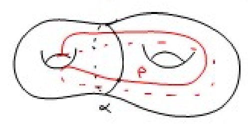
Eg



Surely $i(\alpha, \beta) = 1$ not 0?

Rmk: Algebraic int. # is a lower bound

Eg:



$i(\alpha, \beta) = 4$

"Clearly".

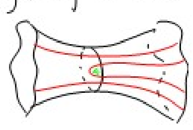
Bigon Criterion:

$i(\alpha, \beta) = |\alpha \cap \beta|$ iff $S \setminus (\alpha \cup \beta)$ has no bigon.

Def: If $S \setminus (\alpha \cup \beta)$ has no bigon say α, β are **tight**.

[NB: Everything is transverse]

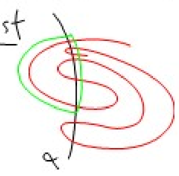
Picture



Thus: $\exists$ bigon $\Rightarrow |\alpha \cap \beta| > i(\alpha, \beta)$.

Note: bigon need not be innermost

Bigon isotopy may decrease $|\alpha \cap \beta|$ by more than 2.



Other direction harder.

Case 1: Suppose $\alpha \simeq \beta$. So we claim $\alpha \cap \beta \neq \emptyset$ iff there is a bigon.

[obvious that $i(\alpha,\beta)=0$; push-off.]

Prove this by: lift to the α-cover of S and apply the annulus case. [Exercise: The α-cover is a non compact annulus]

Also: If $\alpha \simeq \beta$ and $\alpha \cap \beta = \emptyset$ then α, β cobound a product annulus. // case 1.

Case 2: $\alpha \not\simeq \beta$. Suppose that α, α' are isotopic and both tight wrt β.

We will find sequence

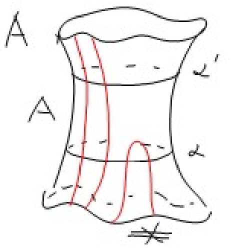
*) $\alpha_0, \alpha_1, \alpha_2 \cdots\cdots \alpha_N$

*) $\alpha = \alpha_0$, $\alpha' = \alpha_N$

*) $|\alpha_i \cap \beta| = |\alpha_{i+1} \cap \beta| \ \forall i = 0, 1, \cdots N-1$

$\Longrightarrow |\alpha \cap \beta| = |\alpha' \cap \beta|$ and proves the criterion.

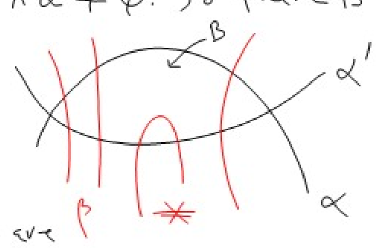
Case 2a) $\alpha \cap \alpha' = \emptyset$. So α, α' cobound annulus, A



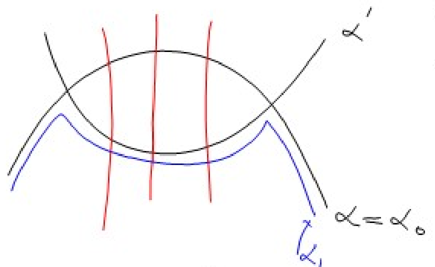
Consider the components of $A \cap \beta$. Use the classification of arcs/loops in A

If there is an iness arc in $\beta \cap A$ then $\alpha \cup \beta$ or $\alpha' \cup \beta$ not tight ⨳. Hence all arcs ess $\Longrightarrow$ There is bijection between $\alpha \cap \beta$ and $\alpha' \cap \beta$. // case 2a.

Case 2b: Suppose $\alpha \cap \alpha' \neq \emptyset$. So there is a bigon between them B



Consider components of $\beta \cap B$. Claim: these are arcs (else done) and they connect opposite sides of B [use $\alpha \cup \beta$, $\alpha' \cup \beta$ tight]



Note that $|\alpha_0 \cap \beta| = |\alpha_1 \cap \beta|$ and $|\alpha_1 \cap \alpha'| < |\alpha_0 \cap \alpha'|$

Induct to reduce to case of $\alpha \cap \alpha' = \emptyset$ // Criterion

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Properties [Exercises]

(i) $i(\alpha,\beta) = i(\beta,\alpha)$ [Symmetry]

(ii) Def if $r \in \mathbb{R}_{\geq 0}$ $i(r\alpha,\beta) = r \cdot i(\alpha,\beta)$.

(iii) $\forall f \in MCG(S)$ $i(\alpha,\beta) = i(f(\alpha), f(\beta))$] MCG invariance

(iv) if T_β = Dehn twist about β $i(\alpha, T_\beta^m(\alpha)) = i(\alpha,\beta)^2 \cdot m$

(v) $\forall \alpha \ \exists \beta$ s.t. $i(\alpha,\beta) \neq 0$

(vi) $\forall \alpha, \alpha' \ \exists \beta$ s.t. $i(\alpha,\beta) \neq i(\alpha',\beta)$ } Non-degenerate